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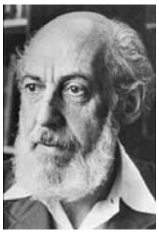
The Mystery of the Black Knight's Noetherian Ring

An investigation into the story-mathematics connection
with a small detour through chess country

A keynote address to the Fields Symposium on Online *mathematical investigation as a narrative experience,* Faculty of Education, University of Western Ontario, 11-13 June 2004



To the memory of Samuel Eilenberg,
co-inventor of Category Theory,
who thought I could become a mathematician.

Because something is happening here
But you are not absolutely certain what it is,
Are you, Monsieur Bourbaki?

Paramathematically adapted from
Bob Dylan's "Ballad of a thin man"

Less is more but it's not enough.

The Guerilla Girls

In a lecture given last year, entitled "Embedding mathematics in the soul: narrative as a tool in mathematics education",[1♣] I tried to summarize the experience and the arguments for using stories as a bridge facilitating students' access to mathematics. None of this was brilliantly original – as it need not be. We know now, after centuries of heated philosophical discussion, that people are much more than logical machines and so it is obvious that the royal road to a young person's brain – also to a not-so-young person's, for that matter -- is through the heart. And even the earliest homo sapiens knew that there's nothing like a good story to siege that most metaphorical of muscle groups. Math stories make people feel closer to math. Elementary, my dear Euclid.

My lecture was inspired by the *embarras de choix* of mathematical narratives that are becoming surprisingly (dangerously?) *à la mode* in recent years. Gone are the times when the Christmas math assignment had to be doing from number *x* to number *y* of the exercise book. Now the students can be given to read an exciting story about mathematics, real or fictional. Gone are the times when the prototype for the mathematician was a short-sighted, flat-footed, ridiculously absent-minded idiot savant. Now, it is Matt Damon and/or Russell Crowe – definite progress!

[♣] Numbers in superscript refer to endnotes, which are mostly bibliographical and/or free-associational. Non-numerical symbols refer to footnotes, which are as a rule parenthetical.

Still, I suspect there is much more to the math-story connection. The suspicion is largely motivated by my personal credo that all art, and especially narrative art, apart from any aesthetic-affective experience it provokes is basically a form of knowledge, an epistemological instrument[2]. And it is in this spirit that I approach the subject of the symposium, being no expert in – though quite an adept practitioner of – online investigation, but having some things to say about the relationship of mathematics to narrative that go beyond finding ways to make mathematics 'student-friendly' or huggable or whatever. And since this is principally a meeting of people separated by a common interest in mathematics education, I must also say this before I get into the subject proper: the reason I will not be talking much about education is because I believe that how we teach mathematics, as a culture, is shaped by how we do mathematics. And so, by addressing the issue of the paradigm shift – if you'll pardon the expression – that is taking place in mathematics in recent years and in which narrative can play a crucial part, I shall also be addressing, though mostly indirectly, the issue of how it can be taught.

♞ ♞ ♞

Being unfortunately an impulsive sort of person, and also being Greek, when a few years back I was invited to talk on the general topic of "mathematics and fiction" I immediately went in search of a general theory. In fact, I concocted a *Gedanken* experiment, to which I gave the somewhat grand title "Euclid's Poetics"[3], the basic idea of which is roughly this: since both a story and a proof share many of the characteristics of a quest (the first both in physical and conceptual space, the second purely in conceptual) it might be possible to establish structural equivalences (isomorphisms) between some of these totally different beings, thus setting up a sort of 'functor' -- to use fancy language -- between the categories of stories and theorems.

In a story, especially a classically constructed one like an epic or a mystery, the quest is the hero's attempt to reach his goal♣. (For Odysseus this would be returning to Ithaca, for Hercule Poirot finding, say, the murderer of Roger Ackroyd.) And a proof is of course also a quest, a logical journey from the premises to the final statement. In "Euclid's Poetics" I suggested that once the very different contexts are forgotten and we map the progress of either the hero or the mathematician as a path in some graph describing a space of possibilities, a proof and a story begin to look uncannily alike – as does an itinerary of a traveller in physical space, but moving in a higher dimensional, abstract space.

The thing to notice here is that both theorem proving and stories are about people in action to achieve a certain task – this is based on the assumption that mathematicians are people, you see. As a rule people have a limited bag of basic tricks to rely on to achieve their tasks, tricks that differ partially (but not totally) from domain to domain and can be augmented or refined by intelligence, imagination and/or courage.

No less an expert on the matter than George Polya gives the following list of tools (tricks) by which mathematicians solve problems:

- Guess and check
- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Work backward
- Use a formula
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use a model
- Use direct reasoning
- Be ingenious
- Solve an equation[4]

♣ Here I look at story types where the quest element is obvious, though the argument can be generalized to any type of story, which can be seen as a quest if we focus on the main characters' movements towards (or away from) their goals – all this is explained in more detail in "Euclid's Poetics".

And anyone who, like the present author, has read enough mysteries can easily dash off a similar list describing a detective's methods to solve a murder case:

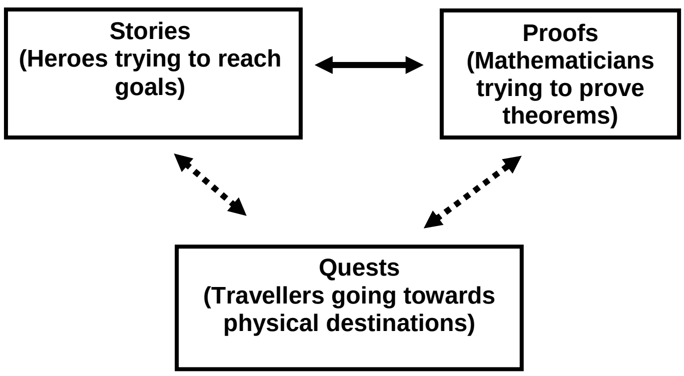
- <u>Guess and check</u>
- <u>Look for a pattern</u> (consult criminal profiler)
- <u>Draw a picture</u> (from witnesses' descriptions, if any)
- <u>Solve a simpler problem</u> (approach the investigation piecemeal)
- <u>Work backward</u> (see who profits by the crime)
- <u>Use a formula</u> (chemical, genetic, of whatever technology is pertinent)
- <u>Make an orderly list</u> (of suspects)
- <u>Eliminate possibilities</u> (through checking alibis, etc.)
- Search for material evidence
- Analyze it well
- Question witnesses
- <u>Use direct reasoning</u>
- <u>Be ingenious</u>

And possibly some more. As you can see, some items (underlined) appear in both lists, some not.

So, if we name the various stops in the two different journeys of discovery, mathematical and investigative, *a, b, c*… and *a', b', c'*… respectively, the stops on the first journey being propositions and those in the second really something of the same (i.e. discoveries of pertinent facts, elimination of suspects, etc.) a graph can be drawn with nodes *a, b, c*… in the first case and *a', b', c'*… in the second. The arrows connecting the nodes describe in each case the trick or syllogism or 'adventure' that the mathematician or the detective employs to move from one to the other. In both cases, the solution of the problem is a path on the graph from some initial point *a* or *a'* to some final destination *z* or *z'*. [5]

We see now that both paths are in some sense *stories*. For the detective this is obvious – at least since Edgar Allan Poe invented the genre. For the mathematician it becomes so if you forget proof as such for a moment and think of *narrating the mathematician's quest to solve the problem.* Obviously this story, as it becomes more and more detailed, requires mathematical background to understand – but that is alright, all stories have a context. At certain points, in a 'high resolution' narrative rendering of reality, the story may become *totally* mathematical, i.e. reducible to a formal language of the type "Let E be a Noetherian ring and *f* a homomorphism $f : E \rightarrow \hat{E}$, where $\hat{E}$ …", etc. But the thing to notice here is that this will happen *often but not always*, i.e. not all parts of the story can be reducible to mathematese. (More on this important point later.)

For clarity, I put the argument of "Euclid's Poetics" in the simple diagram below, saying that we arrive at the solid arrow (functor) on top via the transitive property, combining the two dotted arrows on the lower part. And this roundabout way of connecting proof and story seems handier since the spatial analogy is easier to establish for both[6]:



"General abstract nonsense," you may say♣. Maybe. But now the plot thickens.

I gave the lecture on "Euclid's poetics" in April, 2001. And although it wasn't on April Fool's Day I have to admit that my functor connecting proofs and stories was offered somewhat tongue-in-cheek. Let us say that it was the work more of a writer than a mathematician, i.e. more of an inventor of interesting (hopefully) fictions than someone believing in rigorous proof. But

♣ This is of course how Category Theory is affectionately (?) referred to by some mathematicians.

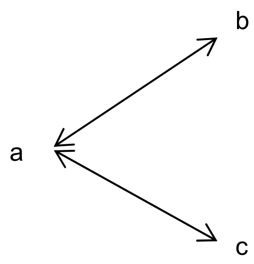
this was done with the best intentions – and by this I mean in the belief that good interesting fictions can be helpful in the real world, if only to spur on a movement that may lead to something true. After all, scientists work with models and almost all models are untrue, at least in the sense of being over-simplifications of the realities they are supposed to portray, i.e. full of lies of omission.

Now, speaking of functors, it is good to remember that the centripetal forces operating in any science, which tend to generalize from similarities (a task usually called induction) are always opposed by legion centrifugal forces, whose task is to identify differences. And in mathematics especially, this opposition has been at the root of most of the last century's epistemological squabbles. The major efforts to find a common unifying language for mathematics, mainly logicism ('all mathematics is reducible to logic'), the set-theoretical approach of Nicolas Bourbaki[7] ('all mathematics is reducible to set theory') as well as the attempt to reformulate the various branches in the Esperanto of Category Theory, have been supplanted by the sins of over-generalization: you see, what a language gains in generality, it loses – at least beyond a certain point – in revelatory power. The problem with *very* general languages (whether Category Theory or Business English) is that they speak economically only at a rather trivial level but rise to ungodly (and thus inhuman) levels of complexity (and pedantry) as soon as they must say something deeper♣. And I make these rather self-evident statements to underline the fact that my main fear in this investigation of affinities between the structure of narrative and mathematical proof is one of triviality. Indeed, I am heeding the words of Michael Atiyah: "The most useful piece of advice I would give to a mathematics student is always to suspect an impressive sounding theorem if it does not have a special case which is *both* simple *and* non-trivial."

And speaking of special cases, let's look at one of proof-story analogy. Take for example the simple path described by St. Basil's dog (see endnote

♣ See the *Principia Mathematica* and the 272 (or whatever) pages it took to prove "1+1=2". For the proof of the infinity of primes it would probably need Borges's Library of Babel.

6) operating in a world where the Principle of the Excluded Middle holds: the graph below describes, in the generic sense, the structure that underlies all possible proofs employing the *reductio ad absurdum*.



Also, it describes all possible detective stories where there are only two suspects to begin with, *b* and *c*, and once *b* is eliminated (through establishing a watertight alibi, say), then *c* is certainly the culprit; also, all love stories where one's true love is discovered after going through all the ravages of being with one's not-so-true love, etc. All of the above are just the path *abac* on the graph.

Having warned myself against saying trivialities – a capital offence in mathematical circles –, I turn now to one of the main areas of focus of this symposium to address "the nature of mathematical experience and what makes for a good mathematics story." Now I may shock you by saying that I think that, from the point of view of good storytelling, "what makes a good mathematics story" is obvious: simply, what makes for a good mathematics story is what makes for a good story of *any kind*, i.e. interesting characters with goals that the reader can identify with, and a path from beginning to end that is zigzagging enough to keep the ride interesting – all this obviously in a context that is in some way mathematical♣.

But the really interesting question is not how mathematics can help create good stories (this being trivial) as *how storytelling can help create good*

♣ The only trick here, of course, is to make a mathematical goal appealing enough for identification. Rather than expand on this I shall state the title of Simon Singh's very popular book about the proof of Fermat's Last Theorem: *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem.* If 'enigma', 'epic quest' and 'world's greatest problem' is a combination good enough to attract us to Indiana Jones or James Bond, the publishers must have thought, then it's good enough for Pierre de Fermat – and so it proved to be.

mathematics – and I hope I shall convince you that this is not so preposterous a task as it sounds at first. And, not to forget our reason for being here, let me say here that I think that any non-trivial answer to that question, however incomplete, will point to ways of applying it to mathematics education which go beyond the “turning-dislike-into-interest” function of mathematical stories.

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Before attempting to answer the question I just posed I shall make a detour, also incidentally throwing light on another element of my title: we have already more than once mentioned “mystery” and we even managed to get a “Noetherian ring” into our discussion. (Please remember, the storyteller’s mentality drives me towards the concrete and exotic sounding.) Now it is time to introduce the “Black Knight”.

I shall refer of course to the game of chess:

In his famous (‘infamous’?) *Mathematician’s Apology*, G.H. Hardy states that “a chess problem is genuine mathematics, but in some way it is ‘trivial’ mathematics.” Now, it is interesting that in this credo of “pure mathematics” a chess *problem*♣ is considered to be mathematics, but not the colossally more complex, and more interesting, analysis of the game of chess itself. Of course, what is characteristic about “chess problems” as opposed to chess itself is their absolutely deterministic character, i.e. the fact that, once found, the solution points to an inescapable situation, a cul-de-sac for the losing side. Obviously, it is this which attracted Hardy, for in this determinism lies what a mathematician usually calls “elegance”[8].

The reason for my detour into the realm of Caissa♥ is that I believe chess is a good platform from which to address the questions of the paradigm

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♣ The term “chess problems”, as used by Hardy, refers to the rather simple combinatorially, and thus of course ‘trivial’ in the mathematical sense, composed (not real-game) situations where one side can mate (only in the chess sense, of course) the other in a prescribed number of moves, usually no more than three. Of course strong chess players find these trivial as well!

♥ A dryad, the – of non-mythological origin – mythological goddess of chess.

shift that is taking place in mathematics in recent decades, from the Platonic-Hardyesque-Bourbakist-EAFist♣ to a messier, more open, less absolutist epistemology. And, what's more, it is a platform from which we can see to greater depth into the math-story link.

There were three reasons that started me thinking along the chess line:

a) The fact that this was one of Alan Turing's first goals when founding computer science: to create a machine that could beat a human player in chess. As everyone knows, this has been realized. And although it doesn't appear likely that a chess program *thinks like* a grandmaster, it can certainly beat one. (Think of the analogy with airplanes: the model for heavier-than-air-flight was the bird, and although airplanes do not flap their wings and could not run for a second on worms and seeds, they are certainly faster than any of their avian prototypes.)

b) My experience of trying to teach chess to my five-year-old son, where I found that the 'story' background of the game (the two warring nations metaphor) was rather useful as an introduction – *but not beyond a certain point.*

c) Reading the fiery Doron Zeilberger's 57th Opinion and his attack on arch-EAFist G.H.Hardy's snubbing of chess as mathematics[9].

As chess-language is infested by military metaphors, I started my investigation by trying to translate chess games into stories. My first attempt started like this:

<u>CHESS STORY #1:</u> *The White Queen sent forth to the field her trusted bodyguard, an excellent scout. The Black King, being a cautious man, sent out a horseman, to inspect his moves from a distance. Fearing some trickery, the White King had one of his Knight's soldiers cover the way for one of his fighting Bishops to approach the field from the side…* [10]

♣ EAFist = adhering to the EAF (Esoteric Abstract Formalist) model; see my "Embedding mathematics in the soul", where this particular mentality is concretely attributed to its three Greek creators, Pythagoras, Plato and Euclid.

Now, this is very pedantic as storytelling and will become worse so, and extremely boring, if the description continues well into the game. And you can imagine that if one would want to make this style totally, unambiguously accurate – which might be altogether impossible – it would become so at the cost of greater length and even worse pedantry. Like that trite piece of popular wisdom comparing women and translations, our story version above will lose in beauty what it gains in accuracy. And even in its most meticulous (and pedantic) it will be a highly uneconomical description of a chess game, certainly much less so than the modern ‘algebraic’ description of the above as: 1. d4 Nf6 2. g3 d5 3. Bg2 e6, etc. And of course we all know that notation is of great importance in the development of thought – after all it was partly notation that did not allow ancient Greek, and then Renaissance, mathematics to progress further than it did.[11]

But as anyone at all familiar with chess knows there are two levels of methods of thinking, strategic and tactical, the second often reduced to its simplest form of calculating or “counting” (moves). This is the thinking of the type “if I take my Bishop to a3 he may take it with his Rook”, i.e. what the *hoi polloi* equate with chess thinking. And, not surprisingly, this is exactly the kind of thinking that computers are excellent at, as they can calculate much faster, much more accurately and to much greater depth than any human being – and without ever blundering!

Here the first similarity with mathematics surfaces: obviously, a big part of mathematical thinking is also purely combinatorial-calculational. And – surprise, surprise! – this also happens to be just the kind of thinking that laypersons *equate* with mathematical thinking. (This is the view by which a mathematician is someone doing complicated sums or solving funny-looking equations on the blackboard.) But we know better: that this is only a *part* of mathematics and one that many good mathematicians dislike – in fact, it often goes against the requirements of elegance and the certain *je-ne-sais-quoi* at the heart of the classical EAF esthetic.

Back to chess. When a player is thinking strategically, he or she is using a very different language from the “if I go there, he goes there” variety. A rather old-fashioned reductionist mentality – which, by the way, would be quite fine were it not for complexity and the truth in that old adage about "quantitative change becoming qualitative at some point" – might insist that any talk of "strategic" is but a mask for our ignorance and our inability to raise the tactical-computational to the absolute level it is entitled to (i.e. being able to calculate *everything*). But for human beings at their present stage of cognitive development, strategic-type thinking is not only useful but necessary – though alas not sufficient for all quests.

Now, a strategic-level rendering-into-story of a game of chess might go like:

> <u>CHESS STORY #2</u>: *The White King marched his best soldiers to the center of the battlefield, but the Black King developed his flanks, preparing against an enemy attack by strong defensive measures. Clearly, the Black King was more intent to protect his kingdom, than to win. Yet, in the end his cautiousness and good sense led to victory. For when the White King made a risky attack, trying to break through to the Black King's camp with a sacrifice of his finest cavalry, the Black King immediately turned the situation to his advantage, crushing his opponent step by methodical step.*

Although the style of Chess Story #2 is not exactly Flaubert either, it certainly goes down better, as story, than Chess Story #1. And also, it tells us things about the game which – though certainly not sufficient to reconstruct it in every detail – are meaningful and significant to a chess player.

If we move one level up, from Kings and strategy to the human players' psychology and method, we get something like:

> <u>CHESS STORY #3:</u> *When Vladimir Kramnik started to prepare for his championship match against Garry Kasparov, he was aware that he would be fighting a stronger opponent, quite possibly the strongest player who ever existed and, what's more, the one with the most profound knowledge of "opening theory" (the variations in the early phases of the game.) So Kramnik, being a very methodical and down-to-earth person, set out to prepare in a way that would neutralize Kasparov's "serve" with white, early in the game, in order to transfer*

the battle to the middlegame, where background research is less useful. He searched with his seconds and analysts in the annals of chess history, which Kasparov knew so well, and found that an old stratagem favored by ex-world champion Emanuel Lasker♣, an outmoded – and thus probably not very well studied by the otherwise very-up-to-date Kasparov – variation of the 'Ruy Lopez' opening, called the 'Berlin Defence' had a lot of hidden potential. So, he studied this in great depth, found many subtle variations, and used it every time he could when playing black. Kasparov was really totally surprised by the employment of an old and long-thought-passive defense which he did not know well enough – nor did he have enough time to study it during the match – and Kramnik managed to get four valuable draws in his games with black, i.e. in the games where Kasparov had the advantage of the first move and could aggressively go for a win. This was crucial in his final victory in the match...♥

Now, this type of narrative thinking (for it *is* narrative and it *is* thinking) is extremely useful, mainly for two reasons: a) as chess is played by real people, human concerns, psychological, sociological or epistemological, play as important a part as the strategic and the tactical, especially in high level encounters; and, b) operating as it does on the level of metaphorical interpretation♦, it is fundamental in building and maintaining the player's functioning at the symbolic-affective level which is so necessary for a winning mentality.

The gist of what we see by our three chess stories is:

- At the very detailed, tactical (and combinatorial-calculational) level, narrative rendering is all but useless, or worse: confusing and irrelevant.

♣ Incidentally, also a great mathematician. In 1905 Lasker introduced the notion of primary ideals and proved his Primary Decomposition Theorem, also known as the Lasker Decomposition Theorem, for ideals of polynomial rings.

♥ It is extremely interesting in our context that the Berlin Defence is not easily amenable to analysis by existing computer programs – so it couldn't be taken apart by Kasparov's staff officers – as it's a more strategic variation, with no pronounced combinatorial features.

♦ I mean this in the sense of it being full of words/concepts derived from the war metaphor: 'fighting', 'opponent', 'tactical', 'stratagem', 'defence', 'surprise', etc.

- At the level of strategy it can offer some useful insights.
- At an even higher level (psychology, grand strategy, etc.) the story level is indispensable, both as a cognitive tool and at the level of motivation, whether positive or negative.

As it is impossible for a player to advance in the game without developing as much as possible his tactical and combinatorial-calculational skills, so it is a serious drawback for him or her who wishes to reach the highest levels to be ignorant of the history of the game and of the life stories of the great players. For chess, to a very serious player, does not begin and end on the board. It is a part of life and extra-chess factors come into it very significantly. As at the strategic level narrative is useful and at the highest level it prevails, storytelling♣ is a necessary part of chess thinking and knowledge. Its importance can be ascertained by glancing through the contents of leading players in their own collections of their best games, where all three levels of analysis are employed. The relative important of each level of analysis differs for each player, and establishes his/her personal style♦. Seen from a higher perspective, the evolution of styles describes the evolution of chess thinking, in its general form.

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I shall return to our chess stories and what they can tell us about storytelling helping us in doing good mathematics. But first I must face the question that begs to be asked: “what, in Archimedes name, *is* good mathematics?”

Well, as an outsider would certainly *not* expect – what with mathematics being ‘the Queen of the Sciences’, the domain of ‘absolute and certain knowledge’ and so on –, professional mathematicians have very differing views on this and their answers would form a highly inconsistent

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♣ By the way, you can substitute ‘narrative discourse’ for ‘storytelling’, if you find the latter too light.

♦ A small example: Kasparov has many more calculations than Karpov.

body of statements. But what is really interesting is that most, if not all, would agree that: a) the distinction between good and bad mathematics is highly meaningful and, b) in a way strangely smacking of Gödel's theorems, the issue cannot be disputed (let alone settled) *within* any particular mathematical theory♥. And what's more – this *unlike* Gödel's theorems – it is extremely difficult to imagine that it can be resolved even at the level of a higher, more inclusive formal theory.

One does not need Gödel's genius to prove the endomathematical insolubility of the problem: after all, no mathematical theory contains a language adequate enough (actually: *vague* enough) in which even to state it. In fact, if the EAF model extended its strict demands from mathematics to thinking or talking *about* mathematics, a member of the totally *purus* species of mathematician, faced with the problem of 'goodness' would have to reply, like a good *entre deux guerres* logical positivist♥: "This is a pseudo-problem, a statement which is not just unprovable – and thus already unacceptable by strictly endomathematical EAF criteria – but unstatable. Quite literally, it is *non*-sense!" Yet, the intriguing thing is that mathematicians as a rule do not give this reply[12].

Hardy goes on and on about mathematical 'beauty' in his *Apology*, a quality he also ascribes to chess problems. But chess problems are "trivial", he says, because good mathematics has to be "*both beautiful and important*". Now, as Hardy couldn't give tuppence for anything as gross as *applications*, by "important" he obviously means "important for mathematics". And though one might try to define this criterion formally – oh, something having to do with generality, measured as the number of deeper results a theorem could lead to –, any attempt at definition would most probably run very soon into a vicious circle: 'It is important because…… it is important…' Good grief!

The story of the impasse that mathematics reached in its navel-gazing "foundational crisis" period exercise has been frequently told: David Hilbert's

♥ It cannot be *formally* decided within the theory, i.e. the language of the theory itself does not suffice to discuss it.
♥ That is, by misinterpreting the concluding statement of Wittgenstein's *Tractatus*: 'What we cannot speak about, we must pass over in silence.'

dream of reconstructing a totally reference-free mathematics, consistent and complete, out of the most basic material, run up against the problems discovered by Kurt Gödel. What has not been given sufficient notice though is the lack of progress in the equally important need for mathematics to talk about itself *informally yet intelligently*. Unlike the foundational tragedy, this other failure has never even been acknowledged as a serious one – in fact, I don't think that the problem has even been really noticed! – and this I guess mostly because of the blinders set in place by epistemological purism. Beliefs die hard, and despite any protestations to the contrary, almost all mathematicians tend to consider the EAF model as *the* epistemological tool to deal with any level of reality, however messy.[♣]

Hilbert's project of a mathematics so well-founded and rigorous that it can be totally derivable by a machine lacking the human frailties of imagination, intuition, etc.,[♥] was – like the tower of Babel – never completed. But still, untouched by Gödel is the fact that to whichever degree of combinatorial depth the Hilbertian ideal algorithm might go, the theorems it would produce would be *correct*. In that sense, they would definitely be mathematics – but *good* mathematics?

The Hilbert machine brings to mind the scenario of a hundred monkeys typing away: given sufficient time, they would produce all of Shakespeare's sonnets. Sure. Of course in olden times the problem would be you'd need a thousand humans and a few million years to extract the sonnets from the mess, though for a computer having the originals in its memory this would be a trivial task – given time. But if the task in question for the monkeys was not to produce all of Shakespeare's sonnets but some *good new sonnets* as well – and this is certainly no more daunting, statistically speaking – and for a computer to recognize them… Well, how could a computer do *that*? A sophisticated language recognition program could take the selection down to 14-line segments that are grammatically and syntactically ok, use acceptable

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[♣] This would explain the phenomenon that many great mathematicians are notoriously naïve when they theorize on matters exomathematical: they are unconsciously applying the EAF model to realities too complex to sustain it.

[♥] His fellow superstar's Henri Poincaré's poignant joke was that this would be like a machine into which the pigs enter from one side and the sausages come out from the other.

words (though Shakespeare didn't always do that) and satisfy the metric and rhyming requirements of the form. But interestingly, setting up a program that would pick out the *good* poems would be probably as complex (or rather: equally impossible) as finding one to locate the *good* theorems out of the combinatorial diarrhea produced by an infinitely ticking away Hilbert machine.

Of course, hardly anyone believes anymore in the total logical determinism dreamed of by Laplace[13]. We now know that at every level of reality there are at least some phenomena that are irreducible to the laws of a lower level. Chemistry is not totally reducible to physics and biology is not totally reducible to chemistry – though there are strong overlaps. Larger-scale thinking is as a rule required by the nature of a larger-scale reality. Of course, a mathematician can still be forgiven to think of mathematics as totally deterministic: it is after all the privilege of mathematics to deny reality, via abstraction, and thus to contain its own criteria of truth. What a mathematician does, after all, is create rules for one-person games that he or she then proceeds to play. The fact that some of these games are inspired by reality and some of their results can be applied to it, functioning like they do as models, does not – for many mathematicians at least – affect mathematics at all[14]. But all absolutist arguments have to be viewed with much more skepticism in the light of what we are learning about complexity.

And I now make the connection with chess: if our need for strategy (i.e. larger scale, non-formal thinking) is just a symptom of our being mental dwarfs complexity- and speed-wise, then all chess will be eventually reduced to calculating every path on a decision tree, and thus one day a computer will be able to play only perfect games. But complex as chess may be (the total number of possible chess games has been estimated to be about 10^{40} times more than the atoms of the universe), it is infinitely simpler than the body of all possible mathematical propositions – and this without taking into account mathematical fields or sub-fields that have not yet been developed. So: if strategic thinking is necessary in the two-person game of chess, why should it not be in the enormously more complex one-person games of mathematicians? After all, it is not the second player that adds the uncertainty making necessary the abandonment of solely deterministic thinking, but the ratio of the known to the unknown. And this is immense in mathematics, regardless of whether we are talking of the not-yet-known-but-in-principle-knowable propositions, or the in-principle-unknowable, the legacy bequeathed to us by Gödel?

♞ ♞ ♞

It is obvious that concepts like ‘beauty’, ‘goodness’, even ‘importance’ cannot be meaningfully discussed strictly within the limits of a formal mathematical theory. But what of ‘truth’ itself? At one level, that of ‘truth values’ of propositions, this is obviously within – indeed at the heart – of any formal theory. But I am thinking of Albert Einstein’s comment to a mathematician: “My job is much more difficult than yours. What you say has to be right – what I say also has to be true.” Could anything of the sort, i.e. a distinction of ‘rightness’ (for which a positive truth value would be enough) and ‘trueness’ be applicable to mathematics itself? ♣

You see, the problem with a great – if not the greatest – percentage of theorems coming out of a Hilbert-type ‘machine’ would not be that they are wrong but that they are *irrelevant* – as no doubt is some of the mathematics being produced today.[15] A decade ago it was calculated that approximately 200,000 new theorems were published every year. And although I’m sure that almost all of them are *right,* having been peer-reviewed, it is hard to believe that a very high percentage of them is a significant addition to knowledge – and a definition of truth must somehow allow it to connect with knowledge, or else of what inherent value is it?

Am I playing with the definition of words? I don’t think so. If concepts like ‘importance’ (still using Hardy’s word), ‘goodness’, ‘beauty’, ‘relevance’, ‘interest’ etc., are not within reach of a formal theory, then we should allow for

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♣ Assume for example that I now invent and propose arbitrary new definitions and an axiom system which is consistent, etc. This is my mathematical theory, my one-person game. Any statements I produce in it the rigorous way are ‘right’. But can they be awarded the accolade of ‘truth’?

the highest of philosophical values, truth, to be also less-than-perfectly formalizable, even when applied to mathematics.

I am treading on dangerous ground here, and I must be careful. I want to make it absolutely clear that I am not siding here with the promoters of such inanities as ‘feminist algebra’, or ‘Aryan cohomology theory’ or whatnot. Even before Andrew Wiles and his proof, a person who seriously argued that whether $a^n + b^n = c^n$ has integer solutions for n>2 is a matter of taste, or race, or socioeconomic factors, would be rightly considered a crank.

Clearly, any theorem established by peer-accepted proof within a given axiomatic system with well-defined rules of inference is ‘right’. But no amount of proving can convince a mathematician to believe that Fermat’s Last Theorem is as important as all the hype tells us it is.[♣] Yet, there is no chance in a zillion that Wiles would have become a front-page celebrity, or the story of his effort a bestseller, if it was simply said of him that he proved the Shimura-Taniyama Conjecture, which is in fact exactly what he did. But to us writers this comes as no surprise: we have learned the hard way that human society feeds much more on the mythological (which is of course narrative) than the rational. And what myth-eater worth the name would want the Shimura-Taniyama conjecture, given also a choice of “the world’s greatest mathematical problem” as FLT became to the media, for a while anyway?

Richard Feynman has said that “mathematicians can prove only trivial theorems, because every theorem that is proved is trivial”. And although this is certainly partly a joke on mathematicians’ use of certain concepts, there is an element of truth in it: known (i.e. proven) mathematics is in a certain sense “obvious”. But that is where formal theories stop. If we move into the sphere of the unknown, the EAF criteria wither. For example, there are many excellent mathematicians who *believe* that the Riemann Hypothesis is true but there are also some, equally excellent, who *believe* it probably is only partly true. And what of all other results of which there is no proof, or even further, those of which even the basic premises have not been defined? What is the value of

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[♣] In fact, Wiles’ real contribution to mathematics was that he proved a *much more general* result, of which FLT was a corollary, as had been previously been proven by Ken Ribet working on a conjecture of Gerhard Frey.

that part of mathematics for which the only formally acceptable criteria cannot be established? And when we say this, we must remember that it is almost certain that infinitely more mathematics is unknown than known. Are we to exile it from our discourse, must we simply relegate it – if we can describe it well enough – to the "to be possibly proven" list?

Von Neumann's warning is certainly to be taken seriously: "As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from 'reality', it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*..." Yet, this is but one central aspect of the problem. We must not become philistines and say that a direct relationship to exomathematical applications is the sole criterion of importance for mathematics. It is basic; but by no means the only one.

So, if not at applicability, where else should we look for the meaning of mathematics, the locus of the valuation of the criteria of 'good' and 'important'? We mentioned earlier that if we try to solve the problem with purely endomathematical criteria it most probably tends to become circuitous. And more than a hundred years of philosophy of mathematics have not really contributed to answering these questions. We do not yet have a score on which of the traditional camps got more points right – though all the talk about formalism vs. logicism vs. intuitionism, etc., has undeniably spurred-on progress in some mathematical fields♣. And today Platonists, formalists, logicists, intuitionists and constructivists alike, all have to take at least partly into account the not-really-philosophical Darwinian-phylogenetic slant that has recently become very (too?) popular when talking about the ontology of human knowledge of any kind. Indeed, anyone but an ultra-fanatical zealot of the old schools would have to admit that if we take the Platonist and the constructivist viewpoints as two extremes, cognitive science teaches us there is at least an element of truth in both. In the Platonist extreme: it is difficult to

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♣ I think that the most impressive case of this is Kurt Gödel's statement that he could not have conceived of his theorems, in the first place, if he was not a committed Platonist in his mathematical philosophy, i.e. if he did not accept independent existence of mathematical truth.

see how the natural numbers or π can somehow *not* be inherent in the nature of the cosmos ('made by the good God', Leopold Kronecker would say). And in the other end, the constructivist: it is impossible not to admit that at least *some* mathematics, oh for example Topos Theory, or Wiles' way to Fermat, or for that matter FORTRAN – which is not more of an artifice than Topos Theory – is at least partly (heavily) molded by the freedom of human creativity.

But although some mathematicians take an interest – by no means all do – in the philosophy of mathematics, formal or not, and many enjoy reading either scholarly or general accounts of the history of mathematics and the biographies of mathematicians, hardly any important mathematicians take these matters really seriously – I mean seriously enough to affect their work – at least while they are in the peak of their careers[16]. To most mathematicians, any discourse about mathematics outside the EAF model is not doing mathematics but talking shop.

♞ ♞ ♞

In fact, the human activity we call mathematics is traditionally considered to be practiced in two ways:

a. The endomathematical of 'mathematics' per se (sometimes also identified as 'pure') in the sense of formal theories. Here abstraction is of the essence, and truth is equivalent to rightness, i.e. defined by what is rigorously derivable from a particular set of axioms. 'Mathematics' defines the only acceptable epistemological criteria for mathematical truth. It is, officially at least, self-contained and self-motivated, sometimes to extent of solipsism. (But solipsism is not-formally definable.)

b. The exomathematical, operating inside or at the limits of the sciences (sometimes also called 'applied mathematics'). Up to the middle of the 20th century, practically the only non-trivial exomathematical interaction of 'mathematics' was with physics and statistics. But now, in addition to these, computer science, biology, meteorology, economics and other fields have developed bridges to 'mathematics'. And apart from using strong mathematical methods themselves, they contribute to the creation of new endomathematical ideas.

Of course, 'mathematics' (as defined in a. above) forms, and most probably will continue to form, the core of the field, the sea into which the various rivers of applications (b.) flow to become 'mathematics' by succumbing to the rigid, inflexible epistemology, abstract, formal and rigorous, that is its distinguishing characteristic. But in the past few decades, some people – modesty prevents me from saying 'some serious people' – feel that the omnipotence of the EAF view should not go unchallenged. More specifically:

c. A new field called 'experimental mathematics' has come into existence. (Not everyone uses the term in the same way.) And although only few of its practitioners go as far as proposing for it a Popperian, natural science-type epistemology♣, the colossal calculating power of computers is supplanting the autocracy of the EAF model, at least in the sense of upsetting the traditional criteria of proof and showing that powerful, deep results do not necessarily have to be elegant[17].

And it is in this same spirit, though in a very different sense, that I recently proposed in "Embedding mathematics in the soul" not so much adding to the above three senses as acknowledging the existence – and, what's more, the *importance* – of a fourth, valid way of dabbling in the wider of mathematical truth:

d. *Paramathematics*[18], i.e. the multidisciplinary (and – correctly – undisciplined) field, lying somewhere in the overlaps of the history of mathematics, mathematical biography, the cognitive psychology of mathematics, the philosophy of mathematics (mostly in its non-

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♣ Which would allow, for example, the Riemann Hypothesis to be accepted as in some way 'true' until a counterexample is found.

formal forms[♥]), the history of ideas, relevant branches of the history and philosophy of other sciences, and so on, whose aim is to discuss the development of mathematics in a non-formal context, mostly in the narrative mode.

The mode of thinking in this case is the narrative because paramathematics does not aim at reaching abstract, irrefutable proofs, as EAF 'mathematics' does, nor at studying mathematical truth as a natural phenomenon (as does a lot of 'experimental mathematics'). Unlike the philosophy of mathematics which operates with its own a priori principles – and thus has little valid interaction with 'mathematics' – or fields like the cognitive psychology of mathematical knowledge, which follow the epistemological model of the natural sciences, paramathematics operates within the criteria of the 'narrative mode of knowing' as defined by Bruner (endnote 2). It accepts human nature as the locus both of the creation and the reception of mathematical knowledge, thus deals both with mathematics and human beings and values, a domain where there cannot be absolute or final knowledge. The field par excellence for paramathematics is the stories of the questions, the methods, the problems and the solutions defining mathematics. In fact, this is already the most central concern of paramathematics, as the budding, half-formed, slowly-finding-its-form literature of the field attests: thinking non-trivially about problems through their history. And anyone who believes in storytelling as a mode of knowing has probably learned the lesson, that to create a good story about something is definitely a way to think non-trivially about it.

What is crucial to realize here – and like a mathematical truth, almost obvious once realized – is that in some sense *the story of the solution of a problem is more important than the solution of the problem itself*. And this is so – to keep things simple – already because of the basic fact that *it contains it*. In other words, the retelling in the rich context which includes biographical, historical, philosophical and other factors, of the story of the attempts at, or

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[♥] We must not forget that a lot of the so-called philosophy of mathematics, as well as all of metamathematics (the field concerned with the study of mathematical theories) are really 'mathematics' in the sense of being founded and operating totally on the EAF model.

the actual solution of a problem – both story and proof, remember, are the putting into a certain order propositions that get you naturally from beginning to end – contributes to the understanding of the solution-as-discovered (which of course is not the same as the solution-as-published) and in this sense contributes valuably to the exploration of the general domain of mathematical truth.

Car-making is not just the cars, and mathematics is not just the theorems. Mathematics is a complex human activity in which the EAF model and its custom-built rigorous epistemology applies only at the top end – or some end, anyway. We have to realize that by studying the problems and the quests for their solutions we are not studying the history of mathematics but actually doing mathematics, though not in the EAF mode. We are dealing with mathematics and achieving a synthesis which can be both interesting and productive in itself and also, occasionally, lead to new and important 'mathematics'[19].

And to those who think that by all this I am trivializing the mathematical method, or distorting its basic premise by reducing hardcore mathematical thinking to storytelling, I remind the three chess stories, the three levels, tactical-computational, strategic, cognitive/psychological/historical. As in chess, so in mathematics storytelling operates at three levels:

a. Although a syllogism may be structured like a story (and in this sense is a story) at the calculational and/or rigorously deductive level narrative is probably of little or no value as an exploratory tool.

b. At the higher, 'strategic' level of understanding, the structure of an argument becomes more important than the nitty-gritty of getting from *a* to *b*, and thus the narrative mode can be more pertinent.

c. At the highest, historical-biographical-epistemological level, storytelling is predominant. It is here, especially, that questions of value can enter and be thought about, narratively, as in no other form of mathematical activity. Let's not forget that mathematical history has not claimed up to this point such depth – it has not been written as belonging to the history of ideas –, exhausting itself mostly

> in mere biography or the chronicling of certain branches. After all, it was often practiced by retired mathematicians, conforming to Hardy's ridiculous belief that older mathematicians are "second-rate minds" (sic) and thus only fit for "exposition, criticism, appreciation" (see the first paragraph of the *Apology*.)

Thus, paramathematics operates differently at different levels of complexity. We must understand that it is alright to be non-rigorous when doing mathematics – *as long as you are not claiming that you are so being*. And, after all, you can only be rigorous at the formal (the lowest, in terms of complexity) level of knowledge.

♞ ♞ ♞

By way of conclusion, and turning full circle to touch base inside the domain of investigation of this symposium, I want to repeat that the application of the paramathematical mode of thinking to mathematics education is quite direct. By defining the learning of mathematics as not just the introduction to the EAF model, which is what we have essentially been doing for a couple of millennia, but *also* as the narrative study of its problems – there is no suggestion of replacement, only of addition to the EAF model –, we are enlarging the scope of the teaching of mathematics (which almost exclusively meant 'mathematics' up to now) by making it a field that is much closer to the concerns of young human beings. And this is because, unlike 'mathematics' itself, the narrative exploration of its quests is made of the same stuff as life-as-lived or, at least, of a stuff much more reminiscent of its complexities and vicissitudes – and no human being is, alas, too young to have a healthy dose of the complexities and vicissitudes of life – and thus is more important, in all senses of the word.

Acknowledgments

I wish to thank Professors Christos Papadimitriou and Doron Zeilberger for reading this paper and making many valuable comments. Also, chess International Grandmaster Nigel Short and International Master Ilias Kourkounakis for inspiring discussions and comments on my excursion through chess country.

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[1] Available online at http://www.apostolosdoxiadis.com/files/essays/embeddingmath.pdf

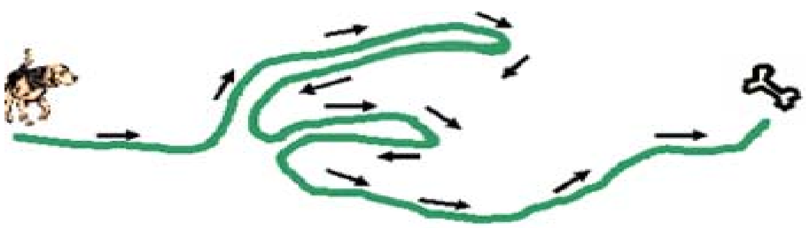
[2] Though I arrived at it through different byways, this view is wonderfully (and famously) put forth in Jerome Bruner's "Two modes of knowing" (In Jerome Bruner, *Actual minds, possible worlds*, Cambridge: Harvard University Press, 1986.) In this article, narrative is convincingly argued to be one of our two basic tools for understanding the world – the other of course being the more acknowledged classificatory-deductive method of science.

[3] Available online at http://www.apostolosdoxiadis.com/files/essays/euclidspoetics.pdf

[4] "George Polya" by A. Motter, at http://www.math.wichita.edu/history/men/polya.html

[5] We notice that for the mathematician proving theorem A, there are really *two* graphs (or, if you want, two paths on the same, more inclusive graph) the first being the course he or she followed to *originally* prove A and the second the – as a rule much more economical and rigorous – course that will be used in presenting the proof in writing, in accordance with the *lege artis*. In the case of the mystery this other path would be the sequence of the logical presentation of the evidence at the trial, obviously totally different from the messy, searching form in which it was discovered. In fact, in each case the first story will be the story of the discovery, while the second will be the story of the crime – an interesting distinction to keep in mind.

[6] The first impetus for these associations came from the unlikeliest source: a theological treatise called *Homilies on the Six Days of Creation* (4th century) where the author, Saint Basil of Cappadocia, propounds the extraordinary idea that mathematicians invented the *reductio ad absurdum* by watching dogs search for food. More precisely, the dog in the diagram below, having smelled a bone in a certain direction will first explore one possible path to it, and if he doesn't find it will not automatically correct his trajectory – if he did that he would have taught us the method of *successive approximations* –, but will go back to the beginning, modify his premise – his course – and try again. (The dog in this particular diagram has never heard of the Rule of the Excluded Middle, so he has to do it twice.)



This is really an amazingly pregnant observation, which – apart from probably being true in the “where does mathematics come from” sense --, links three concepts: a quest story, mathematical investigation, and a course in physical space.

[7] ‘Nicolas Bourbaki’ is the name collectively used by a group of French, mostly, important mathematicians who back in the thirties started the grand project of founding the whole of mathematics on a very refined, formal version of set theory – their opus magnum remains incomplete to this day. What I find amusing in the context of this discussion is that Nicola Bourbaki, i.e. the prime modern advocate of the EAF (Esoteric Abstract Formal) model of mathematics which was founded by the Greeks, mainly Pythagoras, Plato and Euclid, is also in a roundabout way of Greek origin: the name ‘Bourbaki’ originally entered France as that of General Charles Denis Bourbaki, a renowned 19th century military man who was the son of Colonel Vourvachis (Βουρβάχης) a fighter in the Greek war of independence. (Interestingly, Charles Denis Bourbaki was offered – and refused! – the then vacant Greek throne in the 1860’s.)

8 Doron Zeilberger, in a personal communication, says that even in the initial position of a chess game, “the big problem `Can White Win' is equivalent to `Mate in <=200 moves', so it is a ‘chess problem’.” Though of course this is a thought-provoking observation, it does rather extend the accepted usage of the “chess problem”. The theoretically longest-possible chess game has been calculated to last 5,899 moves. But of course, 99.99% of serious games are well into Zeilberger’s ‘under 200’ moves limit.

[9] http://www.math.rutgers.edu/~zeilberg/Opinion57.html

[10] A chess player will see that this is a somewhat messy variation of the start the Catalan Opening.

[11] As late as the 17th century, Pierre de Fermat stated his famous ‘last theorem’ as “cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere”, a statement which would now be “$x^n + y^n = z^n$ has no non-zero integer solutions for $n>2$” – from 156 down to 43 bytes.

[12] I at least have never met a mathematician who did not passionately adhere, whether consciously or not, to both an ethics – if I may so call a discussion of ‘good’ and ‘bad’ – and an esthetics of mathematics. In fact, every mathematician has read and I think most believe in the statement of G. H. Hardy ‘there is no place in the world for ugly mathematics’.

[13] “An intellect which at any given moment knew all the forces that animate nature and the mutual positions of the beings that comprise it, if this intellect were vast enough to submit its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom. For such an intellect nothing would be uncertain; and the future just like the past would be present before his eyes.” Pierre Simon de Laplace, *Philosophical Essay on Probabilities.*

[14] For a taste of the rich discourse on this, see G. H. Hardy’s *A Mathematician’s Apology* (Cambridge University Press, 1940) Eugene Wigner’s famous article “The unreasonable effectiveness of mathematics in the physical sciences” (available at http://www.dartmouth.edu/~matc /MathDrama/reading/Wigner.html); Paul Halmos’ “Applied mathematics is bad mathematics” (1981) in *Mathematics tomorrow* (L.A. Steen, ed.), 9-20, Springer, New York; Doron Zeilberger’s “People who believe that Applied Math is Bad Math are Bad Mathematicians” (http://www.math.rutgers.edu/~zeilberg/Opinion2.html)

[15] I am grateful to Professor Doron Zeilberger for his comment on this point: "(…I think that you) succumb to Hardyian elitism by saying that a significant amount 'of the math produced today is irrelevant', maybe the results are, but the *activity* of producing new theorems is the *message*, the fact that there exists a *culture*, and the whole is much more important than the sum of its parts. So it is nice that there exists a living language called math that is spoken (or rather written) at levels. So that's another analogy with story-telling: There is a place in the world for mediocre and even bad literature/fiction, if nothing else so that the good writers will stick out!"

[16] Field medallists Alain Connes and Timothy Gowers are among the few brilliant exceptions to this.

[17] The most characteristic example of this is the Appel-Haken proof of the Four Color Conjecture, where the solution was derived by reducing all possible maps to 1,936 cases whose 4-colorability was checked by computer.

[18] Paramathematics, from the Greek 'para-' meaning 'at the side of' as in parallel ('at the side of another') or paradox ('at the side of *doxa*, i.e. belief').

[19] When I define paramathematics as, mostly, the 'stories of problems', I do not mean that any account of the story of a problem is of intrinsic value to mathematics. A book such as Singh's on Fermat's Last Theorem, though a great read and a fascinating introduction for the non-mathematical public to a famous problem, does not in any way shed additional light to it. Yet other books – the list is not exhaustive – like Martin Davis's *The Universal Computer*, Peter Pesic's *Abel's Proof*, Amir R. Alexander's *Geometrical Landscapes* and Karl Sabbagh's *The Riemann Hypothesis*, contribute I believe, if not to the advancement then at least to the deepening of our knowledge of mathematics by telling the stories of problems in an interesting way and adding sophisticated 'para-' syllogisms to the formal development. It is worth noting that the author of the last of these books – incidentally republished two years after its original publication as *Dr. Riemann's Zeros* (!!!) – is a documentary producer who studied anthropology and employs in his interviews the approach of a field anthropologist trying to discover the *Weltanschauung* of mathematicians. So: paramathematics is not just for mathematicians, manqué or otherwise!