"The Mystery of the Black Knight's Noetherian Ring"

Comments and Questions from Symposium Participants

Bill Higginson, Queen's University

Let me preface a few questions with a general [read 'ragbag'] statement. This is, in my view, a rich, important and provocative paper for those of us grappling with the very large question of how we might move (in particular bearing in mind the looming prospect of a bag of teaching toys and tricks under extensive augmentation by the potent tools of ICT) the typical/average/representative encounter of learners with mathematics past the bleak, bland, competitive and corrosive experience that it seems to be for all too many of them at present. Having said that, I think that it is not an easy message - or perhaps, more to the point - a collection of messages to generate 'questions' [or implications] from. 'Comments' on the other hand flow relatively easily from the keyboard. Whether this is more a function of the form and content of the paper or of the cognitive style of the pianist [who will, in the tradition of the breed, plead for optimal effort and hence holstered pistols] is, perhaps, itself an open question. I have no doubt, however, given the academic range and linguistic fluency of the paper's author that he will be able to 'counter-comment' with at least as much illumination for the symposium members as he could were he to 'answer questions'. This, and the prospect of dialogical c c-c's [counter counter-comments] are among the conjectured pleasantries for next month's gathering. So, with the reputed advice of Prof. Popper, namely, - the purpose of a lecture is to provoke - duly modified [mutatis mutandis] to fit the cyberized quiz/comment/ reaction form, some consciously pugnacious postulations/pseudo-queries.

Comment/Pseudo-query 1: For mathematical cognoscenti it is nothing less than divine to romp in the Hardyesque/Platonic, EAF vales (veils?) of perfect esoteric/abstract form. For purposes of almost all mathematics education discourse, however, this temptation is, I want to suggest, one to be avoided at all costs. With very few exceptions (the short and not happy era of SMSG in the US - "some math, some garbage" as the more perceptive students were prone to summarize - and Papy's Euro-ventures of about the same period) the problematic reality of contemporary learners of mathematics is not that of uber-rarification, but rather, the ultra-fatigue of T.I.R.E.D. This acronym arose out of a study carried out by the perceptive (Greek) researcher Elena Nardi and her colleagues at the University of East Anglia a few years ago:
A new ESRC-funded report shows that quiet disaffection is ever more evident in the secondary school classroom. In fact students are literally T.I.R.E.D. of maths according to a new profile which includes the characteristics Tedium, Isolation, Rote learning, Elitism and Depersonalisation. http://www.esrc.ac.uk/ESRCContent/news/june02-4.asp

If this is the case, what are the implications for the creators of mathematics learning materials (perhaps online, possibly narrative)?

Comment/Pseudo-query 2:
In our enthusiasm for the potential intellectual/educational benefits of yoking narrative

and mathematics I think it also important to consider potential weaknesses and limitations of this perspective. My thinking in this area as been influenced by a review in the January 10, 2004 issue of *The Guardian* by Galen Strawson (Philosophy, University of Reading) of Jerome Bruner's recent book in this area *Making Stories: Law, Literature, Life* (Ferrar, Strauss and Giroux: 2002).
Strawson's review
http://books.guardian.co.uk/review/story/0,12084,1118942,00.html
is hard hitting. He claims that the application of the idea of creating or inventing 'selves' through 'writing' or 'storying' is one that "has come to dominate vast regions of the humanities and human sciences - in psychology, anthropology, philosophy, sociology, political theory, literary studies, religious studies and psychotherapy". He also thinks that it is nonsense.

I'd like to hear Mr. Doxiadis's views on the issue of weaknesses and limitations of a narrative approach.

Comment/Pseudo-query 3:
Let me be (if I have even a vaguely accurate sense of the term's meaning - more below) aggressively 'paramathemtical' and borrow a phrase from the title of a doctoral dissertation by a 26 year old art critic and historian at the University of Bern in 1907. Wilhelm Worringer's dissertation title was *Abstraction and Empathy* and his subtitle was *Essays in the Psychology of Style.* Almost a hundred years later, Worringer's spectrum seems to be reappearing in a very interesting way - not least with respect to our themes - in the field [wait for it] of paramathematics. This is because researchers like Baron-Cohen at Cambridge in their work on autism are beginning to identify a close link in many cases between mathematics and what he [unfortunately in my view] calls "the extreme male brain". His identified example of such an individual is a Fields medal winning colleague at Cambridge. Quite a lot of the avalanche of paramathematical [there's that word again] 'literature' (using the term broadly to incorporate film and drama) - including Mr. Doxiadis's very fine *Uncle Petros* -dance around the delicate issues of (shall we say) *eccentricity* and *intensity/obsession*. Now this is an area where we need, I think, to be very careful (empathetic?). If we are not, we run the danger of being overwhelmed by the 'oddball' [think of the books on Erdos] image of mathematicians. We do have some very encouraging examples - Nasar's (and Crowe's) *Nash* and Mark Haddon's central character, Christopher John Francis Boone in *The Curious Case of the Dog in the Night-Time*. In this regard I think that it is important to distinguish clearly between the images generated by narratives of 'identified' mathematical figures - for example, Whitmore's *Turing*, Nasar's *Nash* or Kerr's *Newton* and the Matt Damon character in GWH, and the heroines of Stoppard's *Arcadia* and Auburn's *Proof* where the viewers are being encouraged to consider the existence of mathematical talent in previously unexplored areas.

I'd be very interested in hearing Mr. Doxiadis's views [not many authors can boast an endorsement from John Nash on the dangers of mathematical obsession and write plays about Kurt Godel] about this possible set of connections between and among abstraction, empathy (or the lack thereof), personality and narrative.

Comment/Pseudo-query 4:

I've been trying hard to love 'paramathematics' as a term (see above) but it's tough. Perhaps it's a personal foible, but *para* just doesn't do it for me. Part of it is its polymorphous perversity as a profusive prefix [and sometime suffix] - in fields as diverse as chemistry, biology and geography, and accurately or not there are connections in (at least Canadian) English to such dubious partners as 'military', 'normal', 'ochial', 'oxysm', 'site', and 'lysis' [which are not fully balanced in my view by 'good guys' like 'dise' 'bola', 'metric' and 'llel'.] As an early reader of Thomas Kuhn's *Structure of Scientific Revolutions* I may be associating too much with the looseness that I found the term (*Buddy can you ..,..) paradigm* contributing to some not very helpful discussions in the years following the publication of that text. Some readers might - with at least a little justification - take the sense of: 'to ward or defend, to cover from, to shield, to shroud, to shelter' - (as in sol or pluie). Or is that one potential interpretation - Protect yourself from the bright rays - ward off the worst .. well, you (I hope) get the idea.

So, (at least for now), a last question for Mr. Doxiadis - Is he happy with the term 'paramathematical'? I should reiterate that I like the idea quite a lot - I'm just afraid that 'para' will be transformed to 'sub' - in the sense of inferiority - by many people. Has he had reactions of this sort from other individuals? Did he consider other possible terms?

Rob Corless, University of Western Ontario

I will begin with some comments. The first is that the paper has some interesting insights, and I am glad that I have read it, but I have many bones to pick. The first and most central is the treatment of the story metaphor itself. There is no mention whatever of a central element of any story, namely tension, especially erotic tension. Given the prevalence of love-interest in fiction, its absence from a mathematical quest is a glaring difference that (surely) demands comment.

The idea that mathematics is like a simple detective story is less contentious (but less novel) purely because of the mechanical nature of the genre. But, speaking for myself, when I read a detective story I read it for the characters, the manners, the history, the insights---not for the puzzle (the detective, after all, will solve the problem for me). I love Dorothy Sayers, I love Rex Stout, I love Sara Paretsky, not because they can create puzzles that interest me but because they can create believable people, and put them in tense situations, where self-discovery is often the main point. And these are exactly the non-puzzle aspects, those farthest from mathematics.

Yet I deeply enjoy mathematics, too. What I like is the sensation of doors opening, of increased ability, of increased understanding. Surely these are elements of a good story, too?

On a more detailed level, I found that there were many contentious statements in the paper; dogmatic, assertive, and (I believe) wrong. "The royal road to a young person's brain [...] is through the heart". It depends on the person. The thirst for hard knowledge can and does appear in the young, and appeals to the emotions only cloud the issues. " no expert in---though quite an adept practitioner of[...]" Eh? What's an expert, then? "And a proof is of course also a quest" --- well, that one begs the question. To contend: a good story about a quest has tension and peripity (the sudden reversal of fortune from bad to good or good to bad, from Aristotle), both of which have little place in a proof. A good generality, and intellectual force. These are differences that go beyond what is discussed in this paper.

I also believe that the characterisation of mathematics (theorems and proofs) used in the paper is too narrow. There is also experimentation, choice of notation or method or tools, and (most important) the mapping of the theorem to the real world by means of verifiable hypotheses, and interpretable results. These aspects of mathematics would strengthen the connection to a quest.

The central idea of the paper seems to be that story is a good metaphor for higher-level thinking about mathematics (I am not sure what the difference between paramathematics and metamathematics is---what the author calls the EAF model is only a small part of modern mathematics, only about a century old, not millenia as the author claims, and most mathematics happens outside mathematics departments nowadays). But strategic thinking about mathematics is not, itself, mathematics. The EAF says that unless you have control of the details (where costiveness not courage matters) then it's not pure

mathematics. At the higher levels, nonsense may creep in. If any part of the argument is non-mathematical, then the whole is non-mathematical. But this is quite different from stories, where error may well be tolerable. It is quite interesting that this non-EAF mathematics, which is as I stated more nearly the whole of modern mathematics, is closer to the story model than EAF mathematics is.

One statement I found quite funny: "A decade ago it was calculated that approximately 200,000 new theorems were published every year. And although I'm sure that almost all of them are right, having been peer-reviewed[...]"

Really? I am not even sure that the majority of these theorems have been read carefully even by their authors!

But I do agree that most of them are likely useless. How do you even identify the useful ones? Keyword search? Not enough! I once found a very useful and important paper by the coincidence that the author (Gilbert Labelle) used an example that generated the same number sequence as my example; his paper was written in French (not a problem) but also in combinatorics (which was, because I was doing numerical analysis---I would never have thought to look in the European Journal of Combinatorics for a paper on infinitesimal generators when what I was doing was numerical analysis by modified equations).. This agrees with Doxiadis' observation that the Darwinian slant on the ontology of human knowledge is now playing an interesting role (whereas there really are very few plots in use in stories told nowadays).

I once seriously alarmed Tilottama Rajan by suggesting that we need a Critical Theory for Mathematics. She doubted anyone could do it. Doron Zeilberger might begin it, though.

The story of the dog and the wandering scent trail doesn't allow for turbulent diffusion of strands of scent, which allows for much smarter dogs---they can be following a complicated continuous loop of ever-increasing-strength scent, not "returning to where they started".

On the whole, an interesting and provocative paper, even if I disagree with a lot of it. I'll stop there with the comments. My questions are:

1. Quest stories have well-known structures, including things like reversals, frustrations, a mathematical quest of any kind, not just the polished de-scaffolded "Consider X; we assert Y and prove it"?

2. What about "irrelevancies" in stories, as opposed to irrelevancies in story problems or proofs?

3. There are often moral dimensions to stories, which can be examples of behavioural patterns that will enable cultures or memes to survive. Are there moral dimensions to any mathematical fragments? To mathematics as a whole?

Glenn Gordon Smith, … USA

In your paper, you refer to three levels of chess stories with analogs in mathematical story-telling (tactical, strategic, cognitive/psychological/historical). But how should the cognitive/psychological/historical level be further broken down into smaller categories for an even more productive analysis? Your chess story #3, about the Kramnik-Kasparov match (series of games for a championship), is part of a compelling narrative of standard time/plot scale and conflict between two characters. It lends itself to all manner of drama, personality and historical issues. The Fischer-Spassky championship match in Iceland connected to even more drama and world currents (see book "Bobby Fischer goes to War", Edmonds & Eidinow). But chess matches, like sections of mathematicians careers, like well constructed stories, are made up of smaller units (scenes) which are compelling, tightly constructed and educational in their own right.

For example, the *thought process of a chess player choosing a move* is analogous to the soliloquy of a protagonist at the crossroads is analogous to a mathematician solving a new problem. The thought process of a chess player choosing a move has been admirably analyzed by "think-aloud protocol" in the classic paper by Adriaan d. De Groot, "Thought and choice in chess: An overview of a study based on Selzean theory." The study, which included as participants world champions Alekhine and Euwe as well as strong players and amateurs, mentions four phases in deciding on a chess move: 1) orientation, 2) exploration, 3) directed investigations and 4) proof. The soliloquy, voicing ones thoughts while making a pivotal decision ("To be or not to be?"), is fundamental to literature and holds great drama if consequences are terrible or triumphant. How is this building block of narrative translated to the mathematical story? Through think-aloud protocol of expert problem solving? Also, in order to make individual mathematical problem-solving compelling as a story, how does an individual solving a mathematical problem have terrible or triumphant consequences? What would be examples?

Immaculate Namukasa, University of Alberta (graduate student, soon to be a professor at UWO)

1. Would he identify the series of books "Sir Cumference and ... adventures", by Cindy Neuschwander to be good mathematics story books? How and where is paramathematics happening in classroom? I guess I am asking for classroom examples.

2. On page 24 he talks about three levels of mathematical engagement--sylogism, strategic and Narrative levels, how do these relate to each other? What diagram or metaphor, if any, does he use to illuminate this relation? Would he use a diagram like the one on page 6?

Liz deFreitas, University of Prince Edward Island

(1) If our focus is on: "how storytelling can help create good mathematics?" will we impose an instrumentalist vision onto narrative and possibly diminish the power of the text? Similarly, metaphors that are explicitly didactic can give too much closure to an art form. How do we "use" narrative without killing off its power to reach the reader?

(2) There are no formulas for narrative (despite what some how-to manuals suggest) and much of our engagement as readers comes from the play of the language. With regard to the intersection between story and math, is this "play" confined to what you have called "paramathematics"?

Margaret Sinclair, York University

1. You comment that one "can only be rigorous at the formal (the lowest, in terms of complexity) level of knowledge". Could you discuss how the formal relates to early experiences in mathematics, i.e., are there some aspects that are foundational? Is it necessary? Do we build narrative on rigor, or can we/should we build rigor through narrative?

2. Present teachers have been very successful, moderately successful, or unsuccessful at the EAF model. For each group what do you see as the critical interventions that would enable them to understand/adopt elements of the paramathematical field?

Donna Kotsopoulos, University of Western Ontario (PhD student)

The discussion paper prompted me to reread Sfard's (1998) article "The Many Faces of Mathematics: Do Mathematicians and Researchers in Mathematics Education Speak About the Same Thing?". Professional identities, mathematician versus researcher, often form the basis for the valorization of a specific understanding of "good mathematics".

The discussion paper details three levels of narrative inquiry in mathematics : tactical-cognitive/psychological/historical. With rigor only being a contributing factor in the tactical-computational level. For the mathematician, a formalist, rigour is paramount. If mathematical narrative is to be understood as a form of non-trivial thought then how can professional identities that associate "good mathematics" reconcile rigor as being only an elementary component of paramathematics? Would rigor not underscore all three levels? How are the latter levels of narrative inquiry in mathematics assessed ?

Sonja Rowhani, University of Western Ontario (PhD student)

1. I only have a very rudimentary knowledge of chess – maybe comparable to some students' knowledge of mathematics. The conclusions you draw from the three chess stories are diametrically opposed to the conclusions I would draw from them. As someone trying to learn chess, a detailed, tactical story may be more useful than a higher-level story. While for an expert the higher level story may serve as a cognitive tool, it may confuse novices. It seems that the way we read/understand stories is very personal limited by our own understanding. What is your perspective on this?

2. To me it seems that stories in mathematics could succeed in motivating the student and creating an interest in or appreciation for mathematics more so than teaching methods and tools needed to tackle mathematical problems. For example, I am an avid mystery book reader. However, even after reading many stories, I don't think I am qualified or able to solve a crime. Could you delineate the place of stories in mathematics education as you perceive it?

George Gadanidis, University of Western Ontario

1. I find the following comment interesting:

> The reason I will not be talking much about education is because I believe that how we teach mathematics, as a culture, is shaped by how we do mathematics.

Papert in *Mindstorms* says that children enter school as enthusiastic, curious and capable mathematical thinkers – they have to *learn* to be otherwise. Yet how *they* do mathematics is typically ignored (and not reflected) in how we teach mathematics. In fact, some studies comparing the mathematical thinking of grade 2 children with that of grade 4 and grade 5 children (doing identical mathematical tasks) have shown that their mathematical thinking deteriorates, relying more on procedures they do not understand (Kamii; Reid) So the 'we' in the above quote probably does not take them into account.
How might the mathematical thinking of young children (before it is influenced by school or 'well-meaning' adults) help us answer your question of 'what is good mathematics'? Do you see part of the answer in your children?
Also, I wonder if the reverse is also (or more) true: 'how we do mathematics is shaped by how we teach mathematics.' (A chicken-egg problem?)

2. I wonder what would be left if 'story' is taken out of human communication about things that matter. Would we be left with recipes? Stories add meaning and values, something often missing in the unstoried/destoried, recipe-like mathematics we typically experience in school, like:

- when dividing two fractions
- invert the fraction you are dividing by (see reciprocal recipe)
- multiply the two fractions (see multiplying fractions recipe)

Peter Taylor (Queen's University) many years ago visited Bill Barnes' poetry class and noticed that Barnes brought to his class poems he was passionate about. On the other

hand Taylor brought mathematics to his first year Calculus class that did not interest him in the least. Eventually, Taylor and Barnes designed their Math and Poetry course, which they co-taught, and where Taylor brought to class only mathematics that he was passionate about (his mathematical poems).
It seems to me that at the heart of a good math story is a good math problem. But what is a good math problem? I realize that it much depends on how the problems are lived (how they come to life in the classroom), but some problems have more potential than others for leading to a good math story. A simple example in primary school would be the contrast between "what's the answer of 2+2?" vs "the answer is 4 – what was the question?"
So, what in your mind is a good math problem? That is, not what is good mathematics, but what problems lead (or are more likely to lead) to good mathematics?

3. I wonder about the following statement in the context of the education of young children:

> When I define paramathematics as, mostly, the 'stories of problems', I do not mean that any account of the story of a problem is of intrinsic value to mathematics. A book such as Singh's on Fermat's Last Theorem, though a great read and a fascinating introduction for the non-mathematical public to a famous problem, does not in any way shed additional light to it. Yet other books – the list is not exhaustive – like Martin Davis's *The Universal Compute*r, Peter Pesic's *Abel's Proo*f, Amir R. Alexander's *Geometrical Landscapes* and Karl Sabbagh's *The Riemann Hypothesi*s, contribute I believe, if not to the advancement then at least to the deepening of our knowledge of mathematics by telling the stories of problems in an interesting way and adding sophisticated 'para-' syllogisms to the formal development. It is worth noting that the author of the last of these books – incidentally republished two years after its original publication as *Dr. Riemann's Zeros* (!!!) – is a documentary producer who studied anthropology and employs in his interviews the approach of a field anthropologist trying to discover the *Weltanschauung* of mathematicians. So: paramathematics is not just for mathematicians, manqué or otherwise!

I wonder how we do this for young children, without losing sight of the mathematics? It's hard to imagine engaging them with stories about the mathematics of mathematicians. Do we need to write stories about children's mathematics (see also question 1)?
What would work for them? Perhaps considering what math stories you would want to tell your children might be a way of answering this.

4. To summarize, and to exaggerate the argument somewhat, let's consider "Where does mathematics come from?" This is a question we discussed in last year's symposium, in the context of Dissanayke's "Homo Aestheticus: Where Art Comes From And Why." Dissanayake presents elaboration as a process of making special, noticeable. She states, "I suggest then that "elaboration" (another word for "art") is a human need, and that humans evolved to need to be able to show their regard for things that are important to them, and show it *artfully*." Perhaps 'elaboration' may be extended to include the human

desire of making the self more complex, of seeing and shaping experience in new and more sophisticated ways. This does not mean that the art or mathematics created is necessarily more complex, rather that it captures or depicts a greater complexity. Thus, striving to achieve greater complexity is normal, natural and necessary for a human being. In fact, this striving for greater complexity, in art, in mathematics, in science, and in many other areas, is an obvious trend in human history. To use Dissanayake's terms, elaboration is normal, natural and necessary, and art and mathematics are expressions – instances – of the human desire to elaborate.

Interestingly, Dissanayake does not look for the answer to "Where Art Come From" in artists or museum art. Is paramathematics overly concerned with the mathematics of mathematicians? Where else might we see mathematics as "normal, natural and necessary"?

Kamran Sedig, University of Western Ontario

I quite agree with what you say about the three levels of stories (tactical, strategic, meta/grand strategy). For a number of years I have been wondering how to create a connection between all three levels for the purposes of interaction with mathematical representations and their exploration. Do you think all three levels can somehow be represented simultaneously so learners can explore them together and find out how they are related? If so, do you have any thoughts on how this possibility can be operationalized using online mathematical investigation tools?

Christine Suurtamm, University of Ottawa

1.. Would it be possible to teach secondary mathematics in ways that secondary English classes are taught? Students in English classes read great literary works, examine them in detail, look for themes, problems, dilemmas and also work at writing their own stories, poems, and essays. In mathematics, students could read and study great works of mathematics, examine some of the mathematical dilemmas that have been faced and resolved, and also be working at creating their own mathematics proofs and solutions. Would this create a more meaningful program of study?
2.. Is proof really a story? Or is the evolution of the proof the story?
3.. What makes a good mathematical story? Is the mathematics enough?