

Apostolos Doxiadis

“The Mystery of the Black Knight’s Noetherian Ring”

A response to comments and questions

To begin with, I would like to thank everyone who contributed comments and questions, (“c & q”) in response to my paper. Reading them over a few times, I feel that by reacting I can make my thesis more complete and, hopefully, clearer – again, thank you for this opportunity.

I would like to begin by making two *General points*. Without referring to specific c & q’s in them, I believe that they address, in a rather pedantic idiom, a significant range of the issues raised. But since the c & q’s go into some detail and many raise specific points or objections, I would also like to answer those in *Responses to specific c & q’s*.

General point 1

The field that I label as ‘paramathematics’ can be more or less adequately described as “the study of mathematical history and/or biography from a particular point of view”, this point view being a combination of an epistemology and a rhetoric, mixed to varying degrees in each particular case. (Incidentally, I note here that though I ardently promote the need for the field’s existence, I have no very strong feelings concerning my particular choice of name for it – what’s in a name, after all?)

Thus, when the term ‘narrative’ or ‘story’ is used in a mathematical context in my paper, it means more often than not ‘stories of mathematical problems’. In other words, as far as its domain of enquiry is concerned, paramathematics is a full subset of the universe of mathematical history and biography. However, it is the particular point(s) of view that I am advocating that set this approach apart from most, but by no means all, existing historical and/or biographical work in mathematics.

I referred to an *epistemology* and a *rhetoric* as biasing the stance of a paramathematical narrator. The *epistemological* slant is an emphasis on ‘how we know things’ that is informed by the classical concerns of philosophical epistemology as well as the epistemology of science and mathematics, and the more modern insights and techniques of fields within the general area of the cognitive sciences. This can move from the ‘abstract’ end of the spectrum, where the question ‘what is mathematical knowledge’ predominates, to the totally human-centered, involving issues of how human beings learn, discover and/or know mathematics – and these both at the general level but also applied to specific problems or fields. On the other hand, the *rhetorical* concern is not so much influenced by the need to define, discover or create this kind of narrative, as to effectively communicate it. And this element of paramathematics is, obviously, more relevant to issues of teaching. So: at the one end, a paramathematical narrative with strong emphasis on epistemology will concern itself with how and why a particular mathematical ‘story’ developed the way it did; but all or part of the same story can be told – at the other end – with a rhetorical concern, i.e. an emphasis on it being better understood. Obviously, the two aims, epistemological and rhetorical, can often overlap.

The epistemologically-biased paramathematical narratives have to do with the practice of mathematics, with how mathematics is done. The rhetorical cater more to what we may call, also, ‘mathematics education’. I quote from the article on Elena Nardi’s work referred to by Bill Higginson in c &/or q’s: “Accessibility is often the underlying intention when mathematical learning is reduced to an execution of cues and procedures. And yet, devoid of rationale for their use, these procedures often seem mystifying and alienating and are strongly resented by the students.” If we take the word “context” as wider than – and containing -- “rationale”, a rhetorical/paramathematical approach to the history of mathematics is a tool for making the field more meaningful to students, via the creation of context.

One of the central defining characteristics of storytelling is its *concreteness*, and though generic versions of stories do exist (i.e. ‘boy meets girl, boy marries girl, girl abandons boy for other boy’), usually the more

specific a story is – and this in more or less full contrast to the theorems of mathematics, where generality often goes hand in hand with importance –, the more effective it can be. Obviously, the need for injecting concreteness to mathematical education goes all the way back to the simple (but not psychologically trivial) case of teaching young human beings ‘2+3’ through ‘John has two marbles and Mary gives him three more’ – perhaps this is something approaching a universal species-reflex against the abstraction and generalization inherent in the subject, that many (most?) students find such a barrier to the mathematical world. But though problems can be only be made more interesting to a certain extent by ‘concretizing’ them, whether in books or interactive contexts, this kind of injection of the real world into mathematics goes against the very essence of mathematical thinking which, beyond a certain level of complexity, is almost synonymous with abstraction.

Thus, although the ‘concretization’ approach to problems in a sense antagonizes mathematical thinking, a certain way of telling mathematical history and biography (precisely: *paramathematics*) can remain concrete at any level of complexity of the underlying mathematics, referring as it does to a human underlying reality – which has to be concrete, to have existed! The story of the field and its subfields, as well as of the people who created them and the issues involved are always concrete – and there is no limit to the sophistication this discourse can go to. From ‘one plus one is two’ to Noetherian rings, there is a story to tell, a complex story of discovery, that can admit as much rigor and sophistication as we want in its more concretely mathematical arguments.

General point 2

This brings me to the second point: that paramathematics in no way aims at replacing – Euclid forbid! – mathematics, either as a discipline or a form of teaching. I strongly believe it is a *necessary* complement to mathematics though; and, in fact, paramathematics is something mathematicians have been doing all the time on the side of their mathematics,

though they have not spoken about it except very recently. Of course it is in no way *sufficient*, though.

So: what I am really suggesting in my paper is that the acceptance of the fact of a paramathematical intelligence operating on the side of the mathematical, in the mathematician, the teacher and the student, enriches the field and its understanding as it forms its natural *context*, which is of course by itself a “rationale for its use”. A rationale does not have to be of the type “arithmetic it is useful to keep account of your spending” or “calculus is necessary to build rockets/cars/computers...” A rationale for human action needs cerebral justification only when it is not supported emotionally -- one of the ego’s prime defense mechanisms is called, after all, *rationalization*. (This is what is usually meant by a response “if you have to ask, the answer is no”.) And only a storied universe can provide this richness of meaning, necessary to make things acceptable without specific rational explanation. Mathematicians operate, live, are motivated and enriched by this world or paramathematics – i.e. they live inside the ever-developing story of mathematics. How do we expect students and teachers to do mathematics being completely outside it?

Mathematics and paramathematics are different. This is an important truth. But it should not blind us to their areas of overlap. It is precisely for this reason that I speak of the ‘three levels’ of operation of a story in mathematics. (Of course there is a much richer stratification, the three levels marking significant positions on a curve.) And to further clarify this point, I rephrase it: what I am saying, in essence, is that only certain parts of the mathematical process can be fruitfully transferred into stories. At the lower level of complexity, the computational/formal level, this is practically nil. At the other end, the large scale/bird’s eye view/higher/historical, it becomes extremely important. What happens in between depends on the particular slant of a narrative and its function. Sometimes a narrative recounting the progress of a mathematical field with minimal formalism, and even emphasis on non-mathematical criteria, is all important. Sometimes formalist rigor is required. But as extreme mathematical rigor usually enters a field *after* its creation – sometimes with detrimental consequences to the creators, as Andrew Wiles

almost found out when the gap in his original proof of Fermat's Last Theorem was discovered – so mathematicians (and one would think: teachers and students) can talk about and even do mathematics (*sometimes!*) non-rigorously but significantly. And when there is no need for rigor, that is the time for the narrative thinking to creep in.

Obviously, the distinctions operating at the three levels concern the *part* of a paramathematical discussion directly concerned with the mathematics. I mean of course that a paramathematical 'quest story' will certainly contain aspects like the psychological, personal, social, general historical, etc. that – though they may be important to the creation of the mathematics – need not concern themselves in any way with formal arguments, being more about the creators than the creations.

I now move on to:

Responses to specific c & q's

I respond the c & q's in the order in which they were sent to me. Please excuse my omissions, or the rather carefree style, which is really the result of my time pressure – as a chess player might say. For reasons of completeness, I quote at least the part of a comment or question that intrigued me, or requires a direct response [in blue](#). However, to do full justice to the comments and questions as they were posed, I must refer the reader to their full text.

Bill Higginson

[..The problematic reality of contemporary learners of mathematics is not that of uber-rarification, but rather, the ultra-fatigue of T.I.R.E.D. This acronym arose out of a study carried out by the perceptive \(Greek\) researcher Elena Nardi and her colleagues at the University of East Anglia a few years ago: *A new ESRC-funded report shows that quiet disaffection is ever more*](#)

evident in the secondary school classroom. In fact students are literally T.I.R.E.D. of maths according to a new profile which includes the characteristics Tedium, Isolation, Rote learning, Elitism and Depersonalisation.

If this is the case, what are the implications for the creators of mathematics learning materials (perhaps online, possibly narrative)?

I think this is one of the basic points addressed by my main thesis: that all of the five elements of TIRE D can be – if not eliminated – then at least significantly counteracted by an emphasis on context-building paramathematical narratives, setting mathematics in historical, epistemological and human context. The implications for the creators of materials are: try and go back to story, as much as possible. First make them love it, then do it – the other way around does not necessarily work.

I'd like to hear Mr. Doxiadis's views on the issue of weaknesses and limitations of a narrative approach.

Obviously, the narrative approach is not a panacea for anything – especially not in the case of mathematics. It is a valuable tool to work side-by-side with more traditional tools. I hope I am allowed a comment on Galen Strawson's review of Jerome Bruner's *Making stories: Law, Literature, Life* that gave rise to this particular comment:

I had not read the review before Professor Higginson pointed it out in his comment, but I had read Bruner's book and thought very highly of it. Galen Strawson is an analytic philosopher and his view is definitely coloured by the prejudices of that school: a school that delights in branding as 'nonsense' anything that cannot be reduced to facts deducible from primary sense impressions and a formal calculus of elaboration thereof, religion, metaphysics, esthetics, and ethics have been among their targets. And although he is undoubtedly right that a lot of the 'storying' craze is nothing but a (passing, certainly) fad in the social sciences, his objections cannot be generalized to the whole field of narrative inquiry. Of course, fanatical

adherence to any rigid point of view – analytic philosophy included – definitely obscures more truth than it reveals. But to criticize a whole, fruitful and profound approach to certain aspects of human activity because of its excesses, is to throw out the baby with the bathwater.

...Researchers like Baron-Cohen at Cambridge in their work on autism are beginning to identify a close link in many cases between mathematics and what he [unfortunately in my view] calls "the extreme male brain"... Quite a lot of the avalanche of paramathematical [there's that word again] 'literature' (using the term broadly to incorporate film and drama) - including Mr. Doxiadis's very fine *Uncle Petros* - dance around the delicate issues of (shall we say) *eccentricity* and *intensity/obsession*.

I'd be very interested in hearing Mr. Doxiadis's views about this possible set of connections between and among abstraction, empathy (or the lack thereof), personality and narrative.

To me there does seem to be a significant connection between the *eccentricity* and *intensity/obsession* and the creation and understanding of mathematics. Of course, writers and/or directors often stress this aspect for its 'media value' (or whatever): readers and viewers have always been attracted by a good dose of madness in works of biography or fiction. However, the relationship is deeper, going down to an argument that is, roughly: mathematics as an activity is really *systemizing* (by Baron-Cohen's terminology) taken to extremes. To systematize the world you need to simplify it – a process that in mathematics is almost synonymous with abstraction. But as abstraction is to a large degree a Procrustean operation (i.e. one of getting to the essence by discarding huge chunks of reality as 'irrelevant') it is best supported/carried out by people who have a psychological makeup that can relate more easily to a more fragmented – and thus less *balanced* – Weltanschauung. People are not logical machines, except very partially. To operate adequately as such – i.e. with this particular bias to the exclusion of

others – it helps to have a personality structure that can go into a ‘purely intellectual’ mode more easily. As the *Curious incident of the dog in the nighttime* brilliantly demonstrated, this kind of approach, taken to extremes, is highly pathological. Of course, not all mathematicians suffer from autism, Asperger’s Syndrome, paranoid- or obsessive/compulsive personality disorder. But the psychological processes that *in extremis* characterize these afflictions are at least partly essential to mathematics, a field that is abstract, formal, combinatorial, meticulous, occasionally extremely sensitive to the minutest error. It is these parts of our functioning, if distilled and applied, that create a lot – though not all of mathematics. And though we do not have to be overall pathological to be mathematician, the part of us that does mathematics, if it became dominant in a personality, would definitely lead to pathology, to the total lack of that *negative capability* that John Keats defined as so necessary for a strong poetic relationship with the world – this cannot be excluded from any meaningful human life. This is a huge and largely unexplored subject, though. Ben-Ami Scharfstein’s book *The Philosophers: Their Lives and the Nature of their Thought* (Basil Blackwell, 1980) contains some relevant material, especially relating to the people behind the ‘foundational crisis’ of mathematics, in the late 19th and early 20th century. It does try to trace the philosophy and the mathematics back to personality structure, I think quite successfully, and certainly in a thought-provoking way. But this is a huge and extremely interesting topic to the complexity of which the above comments cannot do but a minimum of justice.

Now, as far as empathy is concerned – I understand Professor Higginson’s comment to mean empathy with such human conditions in a mathematical narrative – I think that it comes, as in so many other cases, with understanding. And this, of course, necessitates that we transcend the ‘absent-minded professor’ stereotype, and see in the grandeur of a purely mathematical viewpoint also its tragedy.

So, (at least for now), a last question for Mr. Doxiadis - Is he happy with the term 'paramathematical'? I should reiterate that I like the idea quite a lot - I'm just afraid that 'para' will be transformed to 'sub' - in the sense

of inferiority - by many people. Has he had reactions of this sort from other individuals? Did he consider other possible terms?

As I said, I will not fanatically defend my choice. I am not madly in love with the term 'paramathematics' and I would gladly see it replaced by a better one, if one is proposed. I do see the point about the prefix 'para-' possibly leaving a derogatory aftertaste (due to terms such as 'parapsychology', 'paranormal', 'paramilitary', etc.) and also the point about the more benign uses of it (as in 'paradigm', 'paradise', 'paradox', 'parallel') not fully making up for this. In fact, now that I think of it, I believe that I was conscious of the negative, even the facetious element latent in the term did not escape me when I coined it. But perhaps I purposefully downplayed the concept by my use of name, so as not to anger purist mathematicians by infringing in the Elysian Fields. If the approach contains its own inbuilt warning against taking itself too too seriously, I thought, perhaps the EAF mainstream will accept it more easily, reacting by: "Well it's *para*-mathematics, after all. No harm done in the poor guys having their say..." That kind of thing.

Rob Corless

...There is no mention whatever of a central element of any story, namely tension, especially erotic tension. Given the prevalence of love-interest in fiction, its absence from a mathematical quest is a glaring difference that (surely) demands comment.

The idea that mathematics is like a simple detective story is less contentious (but less novel) purely because of the mechanical nature of the genre. But, speaking for myself, when I read a detective story I read it for the characters, the manners, the history, the insights---not for the puzzle (the detective, after all, will solve the problem for me). I love Dorothy Sayers, I love Rex Stout, I love Sara Paretsky, not because they can create puzzles that interest me but because they can create believable people, and put them in tense situations, where self-discovery is often the main point. And these are exactly the non-puzzle aspects, those farthest from mathematics.

I think there is no contradiction, really. I agree that the erotic element is a central aspect of any story -- at least if extended to metaphorical uses of *eros*, as in *passion*, *obsession*, *affection* which need not be erotic per se to provide a story with an emotional powerhouse. And of course character is a fundamental dimension of story. But both the erotic – in the more general sense – and the attraction of character should be there in (para)mathematical stories. (Most people were attracted to *A Beautiful Mind* – which is not even really paramathematical, just a story about a man who happens to be a mathematician -- purely because of a character's suffering and not out of an interest in the early history of game theory.) And if stories of mathematics, in their educational function, are viewed as *concrete stories of human adventure* all the various elements of good storytelling (of which love interest and character are definitely two) are there. But a mathematical story can have great human interest, even if there is no strong direct love element. If we speak about Galois we have it – but what about poor Paul Erdos?

On a more detailed level, I found that there were many contentious statements in the paper; dogmatic, assertive, and (I believe) wrong. "The royal road to a young person's brain [...] is through the heart". It depends on the person.

Of course it does. But unless the person's character is severely impaired – and even then, though in less direct ways – the language of the emotions cannot be ignored when dealing with a developing human being. Show me one young child who becomes obsessed by a brain-teaser and I'll show you a hundred which are profoundly attracted by a fascinating story.

The thirst for hard knowledge can and does appear in the young, and appeals to the emotions only cloud the issues. (My italics.)

Indeed, but *craving* for hard knowledge is also partly an emotional phenomenon. The very fact that you use the word 'thirst', a metaphorical word, shows that this is no dry 'need for information'. So, I think that rather than clouding the issues, the appeal to the emotions – again, not as a panacea, but as a valid viewpoint – is necessary. (And if you don't like the word emotions, I can say 'the full emotional and cognitive make-up of a

developing human being'.) After all, mathematics education has suffered for centuries from a near-total emphasis on the mechanical. And if this wasn't somehow problematic, we wouldn't be here talking about it.

"No expert in---though quite an adept practitioner of[..." Eh? What's an expert, then? "

I meant I am 'no expert in' in the sense of having no knowledge of the *theoretical* aspects of online investigation, but being a mere experienced user. But I apologize for imprecise word use.

... The central idea of the paper seems to be that story is a good metaphor for higher-level thinking about mathematics (I am not sure what the difference between paramathematics and metamathematics is---what the author calls the EAF model is only a small part of modern mathematics, only about a century old, not millenia as the author claims, and most mathematics happens outside mathematics departments nowadays).

No. One of the central ideas of the paper is that story is a good metaphor for **some** of the higher-level thinking about mathematics. As to the difference between paramathematics and metamathematics: paramathematics has been adequately defined, I think: it is narrative, non-rigorous and non-formal. It does not construct theories, in the axiom-hypothesis-proof model, but it discusses them in a narrative mode. On the other hand, metamathematics is a *type of mathematics*, formal, rigorous, abstract, with mathematical theories as its object. As to the first instance of the EAF model – which I agree is not all-pervasive –, it is quite older than a mere century. It is Euclid's *Elements*.

But strategic thinking about mathematics is not, itself, mathematics.

I agree. But 'nonsense creeps in' as you say only if you brand a non-rigorous narrative a rigorous argument. One should know how one is speaking and at what level. And what kind of epistemological mode one is in, every time. And I am against any such thing. Give to Caesar what is Caesar's.

It is quite interesting that this non-EAF mathematics, which is as I stated more nearly the whole of modern mathematics, is closer to the story model than EAF mathematics is.

I agree that non-EAF mathematics (which is most of mathematics as *discovered and as practised but not as presented and as taught*) is much closer to the story model – if for no other reason because it often works within the human limitations of redundancy and imprecision, and it is an ongoing quest. Hard to formalize, easy to narrate.

One statement I found quite funny: "A decade ago it was calculated that approximately 200,000 new theorems were published every year. And although I'm sure that almost all of them are right, having been peer-reviewed[...]" Really? I am not even sure that the majority of these theorems have been read carefully even by their authors!

Well, obviously you know more about this than I do, and you also have a better sense of humour. So, I modify my statement, to make it more serious, and I hope acceptable: "And although I'm sure *that many of them are right*".

Quest stories have well-known structures, including things like reversals, frustrations, a mathematical quest of any kind, not just the polished de-scaffolded "Consider X; we assert Y and prove it"?

I am not sure what the question is here, but both the de-scaffolded (I like this term!) and the scaffolded mathematical arguments have story-type structures lurking inside them – both stories (and they are *all* quest stories to the extent that they are all decision-making affairs) and proofs have skeletons of complex possible itineraries and interesting actual routes.

What about "irrelevancies" in stories, as opposed to irrelevancies in story problems or proofs?

Obviously in a story the criteria of relevance can be much wider. An element may be irrelevant plot-wise, but relevant from the point of view of character, atmosphere, tempo – whatever. (Usually, in excellent storytelling, a reader or hearer will not mark anything as irrelevant – everything blends in the world of the story.) In a proof, a real irrelevance can be cut out without any cost to the proof. It is usually an expository excess, not a logical one, if it does not harm the main argument. (If it does, it is more likely a mistake.) But in my argument I am mostly speaking about the plot element of story and it is this element which often mimics the logical unfolding of proof. Paramathematical

stories being as a rule based on real stories, their characters are usually given, they are entities to be explored rather than created. And a lot of real 'irrelevancy' may justifiably find its way in a paramathematical narrative, if it serves aspects of the epistemological and/or rhetorical slant.

There are often moral dimensions to stories, which can be examples of behavioural patterns that will enable cultures or memes to survive. Are there moral dimensions to any mathematical fragments? To mathematics as a whole?

A very interesting question. Obviously, looking for an intelligent way to respond to it one would have to consider – among other things -- the sense in which there can be a metaphorical interpretation of a mathematical argument. A naïve and off-the-cuff example: the Fundamental Theorem of Arithmetic could be thus rephrased as “many truths (or problems) can be often reduced to much simpler ones”. I think that doing this transposition, to increase the interplay between mathematics and real life, makes a lot of sense occasionally. And in this sense there is a two-way interaction between mathematics and real life, structural similarities often emerging via metaphor or, in reverse, abstraction. This is also partly true of the relationship of stories to proof. Of course, to go a step further and speak of the 'moral' dimension of a mathematical truth, one would need to refer to a set of values, and this would be quite impossible inside mathematics. But an 'isomorphism' into a world with values, would make it less so.

Glenn Gordon Smith

In your paper, you refer to three levels of chess stories with analogs in mathematical story-telling (tactical, strategic, cognitive/psychological/historical). But how should the cognitive/psychological/historical level be further broken down into smaller categories for an even more productive analysis?

This is a good opportunity to clarify the argument about the story-proof relationship. As mentioned already, the three levels refer really to three characteristic points in a curve. Let's take the case of chess: at the one, low-complexity, end, we have an analysis that is purely combinatorial in nature (a pawn and king ending, say) of which any 'story version' would be totally uneconomical and unenlightening. (One has to remember here that when telling stories – and this point was raised in many of the c & q's – we are *not* speaking purely formally and with merely literal truth in mind. A story operates at least partly metaphorically (that is why *The Old Man and the Sea* is not of interest only to fishermen) and its deeper levels come from- and are addressed to- human beings, not logical machines.) Whatever meaning lurks in a purely combinatorial simple chess situation is ideally expressible practically only in the 'algebraic' language of chess players. But a situation becomes more complex, strategic considerations begin to prevail. Thus it is meaningful to give a general guideline for opening play of the type "a player must try and control as much as possible the centre of the board and develop his/her pieces as much as possible". Obviously, this is a general truth, that *can* often be contradicted, in specific situations, by tactical (or combinatorial) considerations. Yet, its generality brings it much closer to a narrative description of a situation, and a non-rigorous, metaphor-laden language, that comes from- and appeals to- levels of the mind beyond the formal-combinatorial is meaningful here. As we go up, psychological and other factors also become more to the point.

It is pertinent to quote here chess Grandmaster and writer Genna Sosonko: "However, there is a great difference between analysis and the actual process of playing. A game of chess is not a theorem, and the one who wins is by no means always the most logical and consistent, but often the one with the greatest endurance, the one who is the most practical, clever, or simply lucky." (p. 74, *Russian Silhouettes*, New in Chess Editions, Alkmaar, Holland, 2001). Obviously, concepts such as 'endurance', 'practical', 'cleverness' or 'luck' are much more amenable to a narrative than a formal language. So, what really happens is that at any level, a game or a more general conflict, a description of a game that seeks to capture all its depth

must contain both tactical-combinatorial and narrative elements and it is really the relative prevalence of the one or the other that makes me speak of three levels, schematically. But there are really many more.

As an aside here, it is interesting to note the various ways – *levels*, really – at which many important players annotate their own, or others, games: again you may get everything from tactical possibilities and variations – these sometimes discovered *after* the game – to purely psychological or practical considerations. The annotations of ex-World Champion Mikhail Tal are good examples of such multi-layered expansions of the game, often rich in narrative elements. But of course, the narrative element becomes more important if we leave the level of one game and go into a player's overall performance over a period of time. Chess players often study the biographies (and not just the games) of the great players, to learn more about the game.

I like the issue of the 'soliloquy' raised by Dr. Smith very much. In fact, many Shakespearean soliloquies (and 'To be or not to be', the one he mentions, eminently so) are just stops in the action that recapitulate a protagonist's alternatives, and as such are very reminiscent of both mathematical and chess thinking.

But when talking about the thought processes of chess players we must be very careful. A lot of the study of the way of thinking of chess players was historically motivated by the attempt to formalize this process, even before the attempt to model it on a computer. And the study by Adriaan De Groot mentioned is indeed a pioneering study in this direction. But it is important that it has the aim *to show the extent to which chess thinking is a purely logical process*. And an even poor player, like myself, learns to apply this four-step process in analysing situations on the board – but not always successfully, and not always period. There are cases where it is useless.

Yet the study of chess psychology has gone a long way beyond De Groot in understanding how chess thinking *in human beings* also differs from such a formalism. For example, it is known that top player levels have an extremely developed memory and an internalised alphabet of basic game-patterns or sub-patterns (for a grandmaster this can be in the order of tens of

thousands) and, as a rule – this came as a big surprise -- they do not calculate more than average players. Of course, they may *calculate* (this is the word used in chess) at times, and in certain situations. But in others they may be thinking purely strategically, intuitively and even purely psychologically during a game – in fact, the great mathematician and ex-World Champion Emanuel Lasker was known as a very subtle psychological trickster on the board, often executing ‘bad’ moves that he thought would be most effective at that moment for that particular opponent. (An excellent, if slightly dated review of existing research and ideas is *The Psychology of Chess* by W. R. Harston and P. C. Wason, Facts on File Publications, New York, 1983; also important is Pertti Saariluoma’s *Chess Players’s Thinking, A cognitive psychological approach*, Routledge, London, 1995.)

The soliloquy, voicing ones thoughts while making a pivotal decision ("To be or not to be?"), is fundamental to literature and holds great drama if consequences are terrible or triumphant. How is this building block of narrative translated to the mathematical story? Through think-aloud protocol of expert problem solving? Also, in order to make individual mathematical problem-solving compelling as a story, how does an individual solving a mathematical problem have terrible or triumphant consequences? What would be examples?

In a mathematical story choices can be ‘low-level’, i.e. combinatorial and in almost direct rendering of the underlying mathematical argument, but also much more complex, at the ‘higher’ end involving purely non-mathematical concepts such as persistence, endurance, will-power, self-doubt, obstinacy, despair. I can easily imagine such ‘soliloquies’ enriching a description of the mental and psychic turmoil Andrew Wiles went through in his search for the proof of Fermat’s Last Theorem. And the terrible or triumphant consequences are very visible there – at the personal level. Also, if I may, I will mention as example my novel *Uncle Petros and Goldbach’s Conjecture* which contains exactly such soliloquies, as representing the

thoughts and the dilemmas of the protagonist, on which road to take next in his mathematical odyssey.

But such questionings, of self or other, can operate at all levels. When Alice asks the Cheshire Cat which way to go (a low-level, combinatorial piece of advice) the Cat asks where she wants to go – a strategic question. And when Alice says it does not matter where she goes, she gives her a higher level, meta-answer, saying that then it doesn't matter which way she goes!

Immaculate Namukasa

Would he identify the series of books "Sir Cumference and ... adventures", by Cindy Neuschwander to be good mathematics story books? How and where is paramathematics happening in classroom? I guess I am asking for classroom examples.

I have read three Sir Cumference stories with my five-year-old son and he enjoyed them very much – a good sign! Now as to the extent to which these are good *mathematics* story books: well, they are certainly useful in setting abstract concepts in a human context; and indeed they are 'stories of problems'. However, we must keep in mind that these are definitely tales where the rhetorical-didactic element predominates, as they certainly do not add depth to the mathematics in question. And although the mathematics involved, especially in a more clear-cut 'story of a problem' tale, like *Sir Cumference and the Dragon of Pi*, is very close to level 1, the tale does have importance as depicting mathematical method in a way that can make it emotionally interesting. But one does have to keep in mind the age they are addressed to. At that level, the narrative approach of 'making interesting' through a simple idiom works. But it is hard to make this kind of simple, fairytale idiom work for a higher age group.

On page 24 he talks about three levels of mathematical engagement-- syllogism, strategic and Narrative levels, how do these relate to each other?

What diagram or metaphor, if any, does he use to illuminate this relation?

Would he use a diagram like the one on page 6?

This has already been referred to. In essence the level depends on the *extent to which a narrative description of a part of a process – rather than a formal -- could be interesting*. At level 1 this is practically nil, but it rises as we go up. This could of course schematically be represented as an x-y curve, with the level (y) being a function of the importance of a narrative rendering of a mathematical process. Obviously, in a *completed* mathematical process (i.e. a peer-reviewed-and-accepted proof) as in a chess game (or a match, i.e. a series of games) that has *already* been played this can always be described completely formally, if we want – it is theoretically possible to do so. But we talk of a higher level process when a description incorporating narrative is meaningful. But, talking chess for a moment, a purely formal rendering and line of thought cannot be given for a game in progress: too many unknowns. In any mathematical investigation or game (s) the level rises as narrative elements – the more the higher – can add significance to the formal rendition. Of course, schematizing – and, even more, trying to quantify this process – is only done using the language of analogy. The important functions of the narrative process are mostly those that cannot be formalized, referring as they do to higher levels of human functioning.

Liz deFreitas

(1) If our focus is on: "how storytelling can help create good mathematics?" will we impose an instrumentalist vision onto narrative and possibly diminish the power of the text? Similarly, metaphors that are explicitly didactic can give too much closure to an art form. How do we "use" narrative without killing off its power to reach the reader?

Again it depends on the level – and, in fact, this is how the whole issue of levels arose. My point about levels is that there are only some parts of a mathematical process that can be non-trivially rendered by narrative. If the right level is addressed, this should not end up with trivial narratives.

Practically speaking, I think the answer here is to work from real-life, historical

and/or mathematical stories. It is in these that both the mathematics and the narrative can remain interesting at higher levels of sophistication. The Sir Cumference type story can *not*, as a rule, be meaningfully extended to higher levels of mathematical sophistication.

(2) There are no formulas for narrative (despite what some how-to manuals suggest) and much of our engagement as readers comes from the play of the language. With regard to the intersection between story and math, is this "play" confined to what you have called "paramathematics"?

I am afraid I do not exactly understand this question. But I disagree with the generality of the statement 'there are no formulas for narrative'. Indeed, there are no general-use formulas (as, especially, some 'how-to' Hollywood screenwriting books suggest – Syd Field seems to have started this and the approach, and its latest guru is Robert McKee.) guaranteeing excellent results. But there is a long tradition, going all the way back to Aristotle, and progressing to us via Vladimir Propp, the Russian Formalists and the various taxonomies of literary genres (Northrop Frye's 'archetypes of fiction', the Aarne-Thompson classification of fairytale types, etc.) of a more formal study of fiction, where the identification of patterns and underlying flow chart- or graph-like plot structures does yield some significant insights. As a writer, I wholly agree with the point about the importance of language in a narrative. But as Dr. deFreitas says "*much* of the engagement as readers comes from the play of language" (my italics). Yes. *Much* – but not all. Although language is an important element, plot, character, theme – things that can be found in stories of mathematical history and biography – are also very dominant. And I do not believe that in good storytelling the medium *is* the message. It is part of the message, certainly, and it helps us get the message across more profoundly and effectively. But there are extra-linguistic elements in good narrative and it is these I principally address in my discussion.

Margaret Sinclair

1. You comment that one "can only be rigorous at the formal (the lowest, in terms of complexity) level of knowledge". Could you discuss how

the formal relates to early experiences in mathematics, i.e., are there some aspects that are foundational? Is it necessary? Do we build narrative on rigor, or can we/should we build rigor through narrative?

Well, to begin with I must say that most of what I know about early experiences in mathematics derives either from my own, or my children's education – so, do not expect anything profoundly well-informed from me on the side of experience. But as with my discussion of levels, I would say that formalism and rigor are not all-or-nothing phenomena. Obviously, *Sir Cumference and the story of Pi* is not at all formal, while Whitehead and Russell's *Principia Mathematica* is extremely so – but there is a whole range in between. If we look at what we mean when we wonder when a young child's mathematics are rigorous and/or formal, we obviously do not mean the same as we would with a college student of mathematics. A proof produced by a math undergraduate that would be considered sloppy or relying too much on intuition, would be a miracle of rigor for a ten-year-old.

So, what can rigor and/or or formalism mean for a five-, a six-, a seven-year-old? Obviously, at the meaningful-for-this-age point of the low-high rigor scale, I think it would mean something like some dose of precision and abstraction. The demand for higher precision is something students meet increasingly as they grow in most subjects (e.g. history, grammar, geography...) while abstraction is slightly different in mathematics: a grammatical tense takes a certain form and Paris is the capital of France *because that's the way things are* – in these subjects, as in mathematics, it is precise knowledge that is imparted (and insisted on). But in mathematics abstraction goes back to cause. I.e., although we do not teach young children the Peano axioms for arithmetic, and we may encourage them to memorize the multiplication table for reasons of efficiency, only a very insensitive teacher would say to a flabbergasted youngster that $3 \times 5 = 15$ because that's the way things are, or God made it so, rather than explain that what this in fact means is that $5+5+5$, i.e. adding 5 three times is equal to 15, something the child can verify and comprehend from the simpler rules of addition.

So, as a rule, I would say that demands for rigor and precision are (and should be) very elementary for young children, and increase with age. And as

abstraction in some of its meanings or other is probably the most common cause of dislike of mathematics, obviously the less traumatically, and the more gradually, one introduces it the better.

But, interestingly, children do not have a problem accepting formal rules, sometimes even totally absurd ones, in the context of a new game, and this should make us think. The reason probably has something to do with the fact that a game creates a whole mini-world, which is through its story background and/or competitive element much more wholly significant to a child. And the story context, even with very young children, could help them accept the abstract (formal) rules of mathematics much more easily, especially in the context of games. I know that this is also the opinion of Howard Gardner, as mentioned – if I remember correctly, unfortunately I do not have access to the book at this moment – in the last chapter of his book *Extraordinary Minds*.

I know there are many approaches to this, but a narrative approach to problem solving (which might have quite a lot to do with an algorithmic one, relying on the story-quest analogy) would be very beneficial for young children. And this does not mean pure storytelling necessarily but creating story context – as in a game, for example.

Going back to my discussion of levels, I want to remind you that the narrative element becomes less interesting and effective if we go into levels of mathematics where rigor demands (in the childhood sense) are higher. Thus, there is no point in teaching the way to find the square root of a number through a story. This, equivalently, would be like teaching a chess student how to mate with a rook and a bishop. Of course one could say ‘first you cut off the enemy king’s ways of retreat’ (a statement that is also ‘narrative’) but one would assume that a player at the level where he/she can learn this would need no metaphorical ballast to understand it, but would merely apply it on the chessboard.

But if we are talking about a general approach to problem solving, storytelling can be very pertinent. For example, one can see stories that would

embody Polya's principles (listed on page 5 of my paper) on problem-solving very well.

Although talking about 'building narrative through rigor' or 'rigor through narrative' is a little bit like comparing apples and oranges, I have to say that rigor and narrative are *not* opposites. Just different things. There is no reason while rigorous rules and arguments cannot help in the creation of narratives -- *to a certain extent*, of course -- and there is really no reason why narratives could not teach rigor. The exemplary mode is as much a part of fiction, in either direct or indirect forms, as is the cautionary. And one cannot dismiss literature which is close to the exemplary or the cautionary as trivial and didactic. *The Brothers Karamazov*, as most of Dostoyevsky, contain strong elements of both, and *Othello* or Mann's *Doctor Faustus*, are very great literature, despite being very strongly built on the cautionary template.

2. Present teachers have been very successful, moderately successful, or unsuccessful at the EAF model. For each group what do you see as the critical interventions that would enable them to understand/adopt elements of the paramathematical field?

I can say, to start with, that I am pretty sure that the EAF model should not be *forced* on children. Obviously, some children would react better to it than others. (And a mildly autistic ten-year-old might love it -- see the *Curious Incident of the Dog in the Nighttime*.)

Here I want to remind you of Von Neumann's brilliant observation, which is highly significant (if not wholly true), a wise aphorism more than a rigorous truth: "In mathematics you don't understand things. You just get used to them." I think that the truth behind this comment has to do with the fact that the abstract mode is foreign to human ways of knowing and learning and thus needs getting used-to. Of course, I'm sure many mathematicians believe that they *understand* at least certain things in their fields -- but, again, Von Neumann seems to be saying that this 'understanding' is really nothing but the familiarization with the laws of operation of a foreign universe, and this needs time, increased familiarity, experience, co-habitation, the diminution of the initial fear of the strange and incomprehensible... I think it is the same with

children. But while a mathematician exploring a new sub-field (or a pre-med student needing to advance in calculus to get in a good medical school) has an increased motivation to do so, probably extraneous to the field itself, a young child usually has no reason to spend the quality time necessary to “get used to mathematics”. It is precisely in this that the story context is helpful. It would be a very unusual 7-year-old child indeed who, given a choice of listening to a story from a good storyteller (or seeing a favorite DVD) or doing two-figure sums, would systematically choose the latter. Stories are cognitively much closer to life-as-lived than the abstraction of mathematics. This is an obvious point though. What is not so obvious is that mathematics and stories mix.

I think that we must try and see how much of ‘Haeckel’s Law’, from 19th century biology, is applicable to paramathematical thinking. Haeckel said that “ontogeny recapitulates phylogeny”, i.e. the growth stages of an organism, from one cell to full growth in some sense imitate the phylogenetic origin of the particular species to which it belongs. I do not how true contemporary biologists consider this in biology. But obviously it is *to a certain extent* true in mathematics: the stages of mathematical understanding of a growing human being in some way mirror the evolution of mathematical science – I stress that this is useful only as an inspiration, not as an absolute truth. But if taken with an adequate amount of grains of salt, this can be a very interesting guideline for teaching. In fact, it is interesting to note that the whole ‘New Math’ affair was based on a gross violation of exactly this principle: people starting to teach children mathematics with what only very recently in historical terms had been thought to be, by the Bourbakists mostly, the true foundational core of all mathematics, i.e. set theory.

To the other end of the EAF model of mathematics stands the heuristic, messy, calculational, applied one. But the strength of the paramathematical approach is that speaking as it does *of the historical adventure of mathematics as created by people* it can address both models – and both become humanly understandable if viewed within the human story that created them. In fact *Logicomix*, the graphic novel I am working on at this moment with theoretical computer scientist Christos Papadimitriou is the first

(I think!) paramathematical comic book, and recounts, in its first part, the story of the foundational crisis of mathematics, starting from Cantor and ending with Godel's Incompleteness Theorem and Turing's first results. In other words, we use the story medium to speak of the shortcomings of ultra-formalist demands. Of course, as the comics medium is very reader-friendly, the emphasis here is more rhetorical; but the epistemological slant is not at all absent – in fact it contains quite a sophisticated narrative exploration of the relationship of logic to madness.

Donna Kotsopoulos

The discussion paper prompted me to reread Sfard's (1998) article "The Many Faces of Mathematics: Do Mathematicians and Researchers in Mathematics Education Speak About the Same Thing?" Professional identities, mathematician versus researcher, often form the basis for the valorization of a specific understanding of "good mathematics".

You are right. However, as there is no way to discuss valorization of mathematics formally, we either must exclude such discussions from the world of mathematics – which no one does, anyway – or take recourse to another language, one taking account history, to speak intelligently about it. Of course, the problems of valorization are not necessarily ultimately 'solvable'. But most of life is not, anyway, and mathematics, when we are not talking from inside an EAF viewpoint is a part of life.

The discussion paper details three levels of narrative inquiry in mathematics : tactical- cognitive/psychological/historical. With rigor only being a contributing factor in the tactical-computational level. For the mathematician, a formalist, rigour is paramount. If mathematical narrative is to be understood as a form of non-trivial thought then how can professional identities that associate "good mathematics" reconcile rigor as being only an elementary component of paramathematics? Would rigor not underscore all three levels? How are the latter levels of narrative inquiry in mathematics assessed ?

Rigor exists in all three levels. But in the higher levels it does not dominate. This is not an issue, however, of doing 'rigorless' mathematics – it is talking about the sides of issues that are not totally dependent, *at that level of discourse*, on rigor. To give an example, let us assume that we discuss the use of computers in problem solving and we tell the tale of the Four Color Problem also taking into account the Appel-Haken solution, to which many EAF mathematicians will not give the distinction of being a proof. Of course it is rigorous – as any proof has to be, in fact it is *too* rigorous, so rigorous that human beings cannot check it within any reasonable amount of time. The questions related to whether this indeed *is* a proof, as well as what the story of the Four Colour Problem means for how we do and understand mathematics are best told in narrative form, to understand their full complexity. No field of mathematics or metamathematics is equipped to answer the question of whether the Appel-Haken proof is a... proof. But this must be discussed – Paramathematically, I believe. (Robin Wilson's book *Four Colours Suffice* goes some way in this direction.)

At another level, try to imagine a book (not yet written, to my knowledge) introducing topology to adolescents, but going beyond the usual donut/prezel, Möbius strip level. A book referring both to general topology, problems of continuity, the foundational problems those created in analysis, how they were solved in the nineteenth century by Weierstrass and others, then geometric topology, even ideas from algebraic topology, all the way to some understandable low-level truths from advanced modern topology and also applications into problems of space arising out of relativity. Such a book, where the narrative thrust is dominated by the need to know arising from specific problems and crises of knowledge, both intra- and extra-mathematical (as well as intra-domain or intra-psychic in some of its practitioners!) could well give 14-16 year olds a very good idea of how a mathematical field progresses, outlining many of the methodological issues, showing graphs of the way that knowledge progresses, how we generalize from the known, how we form hypotheses, how proofs can be wrong, and so on. All this need not contain but a minimum of formally perfect complete processes or proofs and

could often use very imprecise (*vis-à-vis* rigor) statements. But it would give, if well written, an admirable sense of how mathematics works, and why, and contain a good methodological background. And if there are exercises, they would not have to be topological at all. They might be logical puzzles, generalized logical and methodological techniques used. This would definitely be a non-formal book. But it would be a mighty interesting and useful one.

Sonja Rowhani

1. I only have a very rudimentary knowledge of chess – maybe comparable to some students' knowledge of mathematics. The conclusions you draw from the three chess stories are diametrically opposed to the conclusions I would draw from them. As someone trying to learn chess, a detailed, tactical story may be more useful than a higher-level story. While for an expert the higher level story may serve as a cognitive tool, it may confuse novices. It seems that the way we read/understand stories is very personal limited by our own understanding. What is your perspective on this?

A beginning player that is introduced to chess by an experienced player-coach, will be given equal doses of tactical-combinatorial *and* strategic lessons even from the very beginning. (Strategic lessons are much more general and less formal.) Tactics and 'calculating' offer very near-sighted views of any one point of a game, and as we know to get somewhere it does not suffice to be able to see, or even to have a map, but *we must know where we are going*. Only strategy can do this – and believe me, it is very different from tactics. The more a beginner gets into the world of competitive playing, the more the third level, of a historical and more general understanding of what they are doing becomes necessary to progress. For example, once a player has grasped the rudiments and can play competitively at some decent level, he or she will have begun to acquire and demonstrate a particular *style*, which is very obvious to a more experienced player. Once at this level, it is extremely valuable to study the way great players with a similar style

progressed – not just played in specific games – in their chess careers. Clearly, a lot of chess is purely combinatorial. But not all is.

2. To me it seems that stories in mathematics could succeed in motivating the student and creating an interest in or appreciation for mathematics more so than teaching methods and tools needed to tackle mathematical problems. For example, I am an avid mystery book reader. However, even after reading many stories, I don't think I am qualified or able to solve a crime. Could you delineate the place of stories in mathematics education as you perceive it?

You are right that motivation is one of the primary motives. But I don't agree with you on the uselessness of mystery books in solving a crime. Of course, it would depend on the sub-genre to which the books belong. No amounts of Agatha Christie-reading will make you a Poirot, and that is because they are so little realistic, both from the practical and the psychological viewpoint. But reading good police procedurals and, say, the Inspector Wexford novels of Ruth Rendell, or suchlike more down-to-earth whodunits, would get you very much closer to the mind of an investigator than a complete innocent. And, should you care to, it would teach you quite a lot about investigating and solving some types of real-life problems – not necessarily related to homicide! But, primarily, do not forget that those books are written to entertain. Paramathematical books, as I define them, are not. This is not their primary aim. They are created to investigate mathematical issues and/or to instruct, i.e. with the epistemological and/or rhetorical slants I referred to earlier. Of course, *A Beautiful Mind* or *Good Will Hunting* or *Proof* will teach you nothing about mathematics. But they neither can nor want to. But they are not paramathematical, they are not motivated by a need to understand mathematics, as *Who Killed Roger Ackroyd* is not motivated by a need to understand investigative procedure. But view the full 'CSI' tv-series and read the Patricia Cornwell mysteries... Would you then say that you have

not learned a good deal more than the next person about criminology and laboratory investigation?

George Gadanidis

1. I find the following comment interesting:

“The reason I will not be talking much about education is because I believe that how we teach mathematics, as a culture, is shaped by how we do mathematics.”

Papert in *Mindstorms* says that children enter school as enthusiastic, curious and capable mathematical thinkers – they have to *learn* to be otherwise. Yet how *they* do mathematics is typically ignored (and not reflected) in how we teach mathematics. In fact, some studies comparing the mathematical thinking of grade 2 children with that of grade 4 and grade 5 children (doing identical mathematical tasks) have shown that their mathematical thinking deteriorates, relying more on procedures they do not understand (Kamii; Reid) So the ‘we’ in the above quote probably does not take them into account. How might the mathematical thinking of young children (before it is influenced by school or ‘well-meaning’ adults) help us answer your question of ‘what is good mathematics’? Do you see part of the answer in your children?

Obviously, a question referring to value, such as “what is good mathematics” depends very much on context. And I agree that in the context of education, “what is good mathematics for children” needs to be taken into account – at least to an extent. I definitely see, with my own children, that if you do not want to alienate them, mostly at the pre-school age, it’s better to let them do things (sums, for example) their own way, than sternly impose a method.

But regarding the story approach, creating rich storied contexts for, say, 2-6 year-olds, can unleash a lot of their own creative thinking and potential – more so than a pre-arranged learning conceptual environment. Of

course, in mathematics especially, this type of freedom has to be regulated occasionally, as a lot of mathematics teaching, especially in its emphasis on abstraction and preciseness – I have learned from a previous comment not to use ‘rigor’ uncritically, when talking of children – goes against the natural, free, combinatorial methods of the storyteller in the child.

Also, I wonder if the reverse is also (or more) true: ‘how we do mathematics is shaped by how we teach mathematics.’ (A chicken-egg problem?)

Well, I would say undoubtedly it is a chicken-egg situation. The re-enforcement of the EAF model in mathematical practice certainly comes from EAF-inspired teaching. And, now that you mention it, perhaps it is no accident that the ultra-EAFist Nicholas Bourbaki approach originated in France, a country much admired for the extremely high, purist standards of its mathematical education, especially in the Ecole Normale, from which most early Bourbakists came! Good point! And as with any chicken-egg situation, you can only start to change it at the level of the chicken, it being much more flexible than the egg! I mean, in this case, by intervening in the education, more so than the practice – which mathematicians wouldn’t let you, anyway!

2. I wonder what would be left if ‘story’ is taken out of human communication about things that matter. Would we be left with recipes?

I do not think we would be left with anything – it would be impossibility. Or, if anything at all survived, we would be left with objects. You see, I believe story is inextricable from our fundamental mechanism for conceiving of the flow of the events in our world in a non-chaotic way. The symbolic-linguistic capacity which we possess, as animals don’t, allows us to conceive of and express the world as series of meaningful actions. And in this sense a recipe is ALSO a story, and this is important for our argument.

Peter Taylor (Queen's University) many years ago visited Bill Barnes' poetry class and noticed that Barnes brought to his class poems he was passionate about. On the other hand Taylor brought mathematics to his first year Calculus class that did not interest him in the least. Eventually, Taylor and Barnes designed their Math and Poetry course, which they co-taught, and where Taylor brought to class only mathematics that he was passionate about (his mathematical poems).

Oh, yes, three cheers for subjects we are passionate about – in any case! And of course passion has to do with the emotions (a gross tautological understatement this!) and the emotions are important and stories are ideal carriers for emotions. But because we cannot expect, except in a very utopian environment, the school syllabus to be molded by the teachers' passions, it is good to have stories around anyway, hoping that they embody – if not the teachers' – then at least their creators passions. And good stories certainly do that and have the advantage that the passion is visible and communicable, whereas that is not always so in good mathematics. Newton certainly must have felt passionate about the calculus. But that cannot be seen in the definition of the derivative.

It seems to me that at the heart of a good math story is a good math problem. But what is a good math problem? I realize that it much depends on how the problems are lived (how they come to life in the classroom), but some problems have more potential than others for leading to a good math story. A simple example in primary school would be the contrast between “what's the answer of $2+2$?” vs “the answer is 4 – what was the question?”

So, what in your mind is a good math problem? That is, not what is good mathematics, but what problems lead (or are more likely to lead) to good mathematics?

Perhaps because of my relative inexperience with these ideas, I tend to distinguish two kinds of paramathematical stories – whereas deep down they are one. The two kinds are those of a created quest environment (for younger children) and those where the quest environment comes from the world and history of mathematics itself. And although I am a parent (three times over) I

have very little experience of teaching kids – so I have thought much more and can talk much more intelligently (I hope!) about the second case, the paramathematical approach to the world of mathematics, which would definitely not be ideal for very young children. (The NUMBER DEVIL by Hans Magnus Enzensberger is obviously a good case in point, for the first category.) But, as I said, this is perhaps due to my inexperience in the sense that the same rules apply in both cases. But a good story is a good story – period. Thus, if you are looking for the mathematics that would create a good story, you should look for the mathematics that would support a story with interesting characters, fascinating dilemmas, an engaging story world, strong reversals of fortune – what you would look for in any story.

My only warning to anyone attempting to write and or design such stories and environments would be to be aware of the ‘gear-shift’ factor which I mention in my comments to Kamran Sedig, below.

3. I wonder about the following statement in the context of the education of young children:

“When I define paramathematics as, mostly, the ‘stories of problems’, I do not mean that any account of the story of a problem is of intrinsic value to mathematics. A book such as Singh’s on Fermat’s Last Theorem, though a great read and a fascinating introduction for the non-mathematical public to a famous problem, does not in any way shed additional light to it. Yet other books – the list is not exhaustive – like Martin Davis’s *The Universal Computer*, Peter Pesic’s *Abel’s Proof*, Amir R. Alexander’s *Geometrical Landscapes* and Karl Sabbagh’s *The Riemann Hypothesis*, contribute I believe, if not to the advancement then at least to the deepening of our knowledge of mathematics by telling the stories of problems in an interesting way and adding sophisticated ‘para-’ syllogisms to the formal development. It is worth noting that the author of the last of these books – incidentally republished two years after its original publication as *Dr. Riemann’s Zeros* (!!!) – is a documentary producer who studied anthropology and employs in his interviews the approach of a field anthropologist trying to discover the

Weltanschauung of mathematicians. So: paramathematics is not just for mathematicians, manqué or otherwise!”

I wonder how we do this for young children, without losing sight of the mathematics? It’s hard to imagine engaging them with stories about the mathematics of mathematicians. Do we need to write stories about children’s mathematics (see also question 1)?

What would work for them? Perhaps considering what math stories you would want to tell your children might be a way of answering this.

Ay, there’s the rub! As I said earlier, I have not thought so much about this, as about the ‘stories about mathematicians’, which can certainly be made fascinating for adolescents. Let me give you some very off the cuff ideas on working with younger students.

- a. It is more likely that they should be stories of quests in fictional environments, rather than the world of mathematics.
- b. Still, the world of mathematics and mathematicians should be gently introduced, the figure of a mathematician as a fictional, or half-fictional hero.
- c. Because of the problem with ‘gear shifts’ (i.e. people liking to know what they are reading/what genre they are involved in and not liking to be radically subverted) I would not include too much ‘low level’ computational stuff in a higher level story. But a rather intricate storytelling sequence can be made solely at the low-level. Rule of thumb: if you want to address ‘low-level’ issues, make the story line and environment very lean. If you want a rich story context, don’t do too much low level work.
- d. It’s always a good idea (and whodunits are the model for this) to assimilate methodological and epistemological knowledge into the story – tis much easier to do, seamlessly, than the mathematics. In addition to using stories for teaching skills like sums, operations, fractions, etc, try and do ‘mathematical’ stories for early-age students which are really logical, i.e. they teach them the methodology and tasks of a logical search. Sherlock Holmes is always referring to his methods and epistemology, there is no reason why we could not do this with logic and children. (Some of Smullyan’s books are

great in this, and especially his chess books – as models.) Try to do stories that make of a child a logician – and perhaps show how that can be applied to mathematics, occasionally and softly. A good hero: St. Basil's Hound – the one who invented the reduction ad absurdum!

4. To summarize, and to exaggerate the argument somewhat, let's consider "Where does mathematics come from?" This is a question we discussed in last year's symposium, in the context of Dissanayake's "*Homo Aestheticus: Where Art Comes From And Why.*" Dissanayake presents elaboration as a process of making special, noticeable. She states, "I suggest then that "elaboration" (another word for "art") is a human need, and that humans evolved to need to be able to show their regard for things that are important to them, and show it *artfully.*" Perhaps 'elaboration' may be extended to include the human desire of making the self more complex, of seeing and shaping experience in new and more sophisticated ways. This does not mean that the art or mathematics created is necessarily more complex, rather that it captures or depicts a greater complexity. Thus, striving to achieve greater complexity is normal, natural and necessary for a human being. In fact, this striving for greater complexity, in art, in mathematics, in science, and in many other areas, is an obvious trend in human history. To use Dissanayake's terms, elaboration is normal, natural and necessary, and art and mathematics are expressions – instances – of the human desire to elaborate.

Interestingly, Dissanayake does not look for the answer to "Where Art Come From" in artists or museum art. Is paramathematics overly concerned with the mathematics of mathematicians? Where else might we see mathematics as "normal, natural and necessary"?

I think the analogy is interesting. I am an avid fan of Dissanayake, when she talks – as she almost always does – about art. But I think mathematics is slightly different. It (may) be 'normal, natural and necessary' at

a low level, a level where the complexities of mathematics and logic (and thus the complexities of the human adventure lurking behind them) are rather simple. The problem is that it is rarely fascinating at that level – unlike art, where a simple sonnet, Beethoven's *Fur Elise*, a Mayan sculpture or an Italian fairytale can all be great art. But not so in mathematics. There is no poetry in simple fractions or sums, or the Pythagorean – or if there is, again it is of the 'Mary had a little lamb' variety, comparatively speaking. (But even simple logical puzzles can be fascinating – and logic is a part of mathematics!!!!) So, we need stories to make accessible to the soul what is mentally complex, to create that bridge. The naturalness of mathematics (to the small extent that it exists) is useful at a very low level. If we want to show the naturalness of logic, as a part of the mathematical process, that may be simpler: stories are more logic-friendly than math-friendly.

Kamran Sediq

I quite agree with what you say about the three levels of stories (tactical, strategic, meta/grand strategy). For a number of years I have been wondering how to create a connection between all three levels for the purposes of interaction with mathematical representations and their exploration. Do you think all three levels can somehow be represented simultaneously so learners can explore them together and find out how they are related? If so, do you have any thoughts on how this possibility can be operationalized using online mathematical investigation tools?

Let me for a moment relate my experience while writing my book *Uncle Petros and Goldbach's Conjecture*. Of course, this was addressed to readers of fiction, and I did *not* assume the average reader would be interested in mathematics – quite the contrary, in fact. (But that is also a good assumption if we are addressing the average primary or secondary school student!) What I did want to have there – without yet being conscious of the precise notion of levels – was really to have all three. Starting from the top, I definitely wanted to show how history (Gödel's theorem), society (the unwritten laws of

academia), psychology (Petros's character) interacted with the mind of a researcher and how they influenced his career. For that narrative suited me perfectly, and a lot of the positive reactions to the book from the mathematical world have been variations on the theme 'manages to show how a research mathematician works', etc. And I think this is interesting, both as an epistemological and rhetorical task. Going to the middle (strategic) level, I again tried, I think somewhat successfully, to give an idea of the type of thinking involved when a mathematician tackles a problem, e.g. by using the reductio, by simplifying the problem, by looking at it from a particular side with more developed techniques (the partition problem) etc. And again, this could be done more or less 'narratively', where a good part of this could be the internalised, 'soliloquy' type of approach mentioned by Dr. Smith. As for the lowest level, the 'tactical', or more directly computational, I thought there would not be much sense in putting in stuff of this kind. I had an instinctual sense that if it can't be 'narrativized', then it stays out. Thus, although, for example, I included at first the full proof of Euclid's infinity of the primes, as Petros explained it to the nephew (it's just three or four lines of course) I then cut it out – along with some similar stuff.

But had I been addressing a reader more benevolently inclined to mathematics – say, in a non-fiction work of Singh's *Fermat's Last Theorem* variety --, I would of course go a bit further.

Now, speaking as a writer, I know that the reader's intention in picking up a book is crucial. And though I try to be reader-friendly at all times, and would not be interested in addressing a reader with the declared perversion of liking "difficult books" in literature (i.e. a self-proclaimed masochist) I of course understand that the author of a mathematical treatise can expect a bit more of a willingness to take pains from a reader than the author of a commercial 'page-turner'. And a paramathematical text does not really as a rule expect to succeed, as a rule, by springing on an unsuspecting tourist looking for a long flight's fun read and enslaving his heart to the Queen of the Sciences.

Still, how much 'low-level' arguments can we put into a narrative text? To me, as a storyteller, the operative word here would be 'seamless'. If you are writing fiction (of the not intentionally 'difficult' variety) you do not like your

reader to have the feeling he/she is changing gears all the time – or if so, you look out for abrupt changes. If they aim at shock value, that may be alright, but as a rule you go for an underlying unity of style. Thus, although you can use two narratives, a present-tense and a flash-back, and move quite easily from one to the other once the convention is established or, again, you can use two very distinct and different narrative voices (say a university professor and an illiterate and not-very-bright wino) and again achieve an easy co-existence, you cannot easily say “now you are reading a story / now you are solving an equation / now you are reading a story / now you are doing an integral”. The two don’t mix very well – even if the reader is fluent in both literary and mathematical reading. In fact, that is one of the main problems, in my mind, of that old classic, E. T. Bell’s *Men of mathematics* – when I was in my math reading mood, I’d rather read more hard-core (and more up-to-date, symbolism-wise) versions of the achievements of his heroes, and when I was in my narrative-thirst, hero-worshipful mood I found the mathematics distracting.

Thus, in conclusion, I’d say that paramathematical thinking and writing should tend to stay closer to levels two or three – trying to leave level one, unless for the purposes of illustration or explanation, if not alright out, then at a marginal level.

But some excellent paramathematical works, such as Peter Tasic’s *Abel’s Proof* (also of great interest is his *Mathematics and the Roots of Postmodern Thought*) Leo Corry’s wonderful *The Origins of Eternal Truth in Mathematics: Hilbert to Bourbaki and Beyond* (available on the internet, from his web page), Alain Connes two books of conversations, some of Greg Chaitin’s easier lectures – all works with great epistemological interest – are practically, as a publisher might advertise, ‘formula free’. And Bob Osserman’s great expository – and not only – work, *The Poetry of the Universe* is also of this kind. But look at David Foster Wallace’s *Everything and more: a compact history of infinity*. The amount of ‘hard’ mathematics in it makes it impossible to a general reader to read – and the reviews I read from mathematicians found his arguments totally unconvincing. (It helps if paramathematical writers know some mathematics!) It is a well-tested truism, that people with profound

understanding of a field can usually speak about it in very simple, jargon-free language, and part of the importance of paramathematics has to do with this, the going for the wisdom rather than the knowledge, and the knowledge rather than the information. Of course (as in chess too – but less so) in mathematics often this is inextricable from the formal, combinatorial/calculational work. But not *a/ways* – and paramathematics should tend to stay in that other dimension, as the works stated above (but one) brilliantly do.

So, to try and apply this to the online world: again, there is the difficulty of ‘changing gears’. I find that players of online games, on the whole, like to operate at a certain level and do not like huge jarring changes. Thus, if a user is happily clicking away, he/she will not like to suddenly have to stop and read a twenty-page essay. I think this should be the main criterion online, not so much a narrowness of levels, as a new criterion of what types of thinking, whether narrative or lower-level, can be given a more or less homogeneous hypertextual style.

Christine Suurtamm

1.. Would it be possible to teach secondary mathematics in ways that secondary English classes are taught? Students in English classes read great literary works, examine them in detail, look for themes, problems, dilemmas and also work at writing their own stories, poems, and essays. In mathematics, students could read and study great works of mathematics, examine some of the mathematical dilemmas that have been faced and resolved, and also be working at creating their own mathematics proofs and solutions. Would this create a more meaningful program of study?

I am all for the underlying idea in this, were it not for one problem: the texts themselves. The older ones (Euclid, Archimedes, Pascal, Descartes and Newton, say) which might be more approachable mathematically would be extremely tedious, if not impossible to read due to the antiquated notation. And the closer to us we come in time, the mathematics becomes too complex

to be understandable in the original texts. But on the other hand, it is ridiculous: a student of English has come in contact with Shakespeare, Dickens, Emily Dickinson, Melville, Eliot, Joyce and whatnot, and a student of mathematics thinks humanity's achievement in this field is second degree equations and trigonometry, the equivalents of 'Mary had a little lamb', in a way. And this is a shame – no wonder students don't take mathematics seriously and are not fascinated by it. So, the challenge here is to create the right text: books introducing mathematical history but with the paramathematical, sophisticated slant, for various age groups. To me, there would be no more ambitious project in mathematics education work, and no better spent money by a well-meaning foundation, than to invest in the creation of such texts. I'd love – and may eventually get to – work on one of them myself, though I cannot guarantee the results of course.

2.. Is proof really a story? Or is the evolution of the proof the story?

To speak somewhat schematically: the proof is the 'bare bones', the x-ray of the story. Yes, the proof itself is in a sense the story, but it is too dense to be narratively interesting, and too clean cut too. Obviously, as such, it has no great human interest. To acquire such, two things must happen: a) the human stories of the creators (I don't mean their love-affairs, I mean their dramas around finding the proof) must enter the scene and, b) the proof must also be seen as an epistemological advance, in context, and also partly through its metaphoric significance, in areas of knowledge other than that in which it strictly appears. (The metaphors can also be endo-mathematical of course, too, generalizing results to other fields, or extracting the strategic methodology.)

3.. What makes a good mathematical story? Is the mathematics enough?

The prevailing wisdom in the world of fiction is that you can make a good story out of anything. And I think in some sense the same would be true in mathematics, as regards the underlying logic. But of course, the added, human dimensions (the story of the discovery, the importance, other important

factors in the historical, epistemological, psychological or social sense) count much more for its rhetorical potential.